Electromagnetic Propagation and Scatter for Mixed Paths with Multiple and Finite Discontinuities and Application to Remote Sensing of Sea Ice with HF Radar

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ABSTRACT

A general formulation of the problem of determining the electromagnetic field resulting from a finite electric current source located above a plane surface is presented. The plane surface is assumed to have electrical properties subject to variation in one direction. The analytic technique used has the following properties: the source or exciting field is arbitrary; the resulting equations directly relate the unknown electric field intensity and the source field intensity; and boundary conditions evolve directly from the formulation in the form of auxiliary equations. The technique has also been used successfully in the treatment of rough surface propagation and scatter and layered media propagation.

Solutions of the problem for the electric surface field are given for the following special cases:

(1) a surface with one simple discontinuity,

(2) a surface with two simple discontinuities, and

(3) a surface with a discontinuity involving a finite region.

Expressions for the electric field in all regions of the surface are given in each case. These expressions are used to develop radar range equations and radar cross sections for surface wave radars.

Applications of the theory to sense sea ice and icebergs remotely using HF radars are discussed. Several examples of predicted return from iceberg models are given and the magnitude of the return related to ocean clutter.

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TABLE OF CONTENTS

	Page
ABSTRACT	i
ACKNOWLEDGEMENTS	ii
TABLE OF CONTENTS	iii
LIST OF FIGURES	v
1.0 INTRODUCTION	1
2.0 PROBLEM FORMULATION	5
2.1 Introduction	5
2.2 Maxwell's Equations for the Complete Space	6
2.3 The Basic Equation for the Electric Field	7
2.4 The Field Decomposition	12
2.5 Reduction to Integral Equations in the Case of a Simple Discontinuity	14
3.0 APPROXIMATE SOLUTIONS FOR THE ELECTRIC FIELD IN THE CASE OF SURFACES WITH SIMPLE DISCONTIUITIES	29
3.1 Simplifying Assumptions and Resulting Equations	29
3.2 A Series Solution for the Electric Surface Field	39
3.3 The First Order Forward Propagated and Backscattered Vertical Electric Field	42
3.4 Higher Order Terms for the Forward Propagated Vertical Electric Field	51
3.5 The Electric Field for the Case of Two Simple Discontinuities	55
3.6 The Electric Field for the Case of a Discontinuity Involving a Finite Region	64
4.0 ASYMPTOTIC EXPANSIONS AND RADAR CROSS SECTIONS	71
4.1 The Backscattered Field of a Simple Discontinuity: The Monostatic Case	71

4.2 The Backscattered Field for the Case of Two Simple Discontinuities: The Monostatic Case 76
4.3 The Backscattered Field for the Case of a Discontinuity Involving a Finite Region: the Monostatic Case 80
4.4 Radar Cross Sections 83
4.4.1 Radar Range Equation and Radar Cross Section for a Simple Discontinuity: The Monostatic Case 83
4.4.2 Radar Range Equation and Radar Cross Section for Two Simple Discontinuities: The Monostatic Case 85
4.4.3 Radar Range Equation and Radar Cross Section for the Case of a Discontinuity Involving a Finite Region: The Monostatic case 86

5.0 CONCLUSIONS 89

5.1 Discussion 89
5.2 Applications to Predictions of Returned Power and Radar Cross Sections for Sea Ice Targets 90

REFERENCES 93

APPENDIX: COMPUTER PROGRAM FOR THE CALCULATION OF RADAR RETURN FROM FINITE IRREGULAR DISCONTINUITIES 95

LIST OF FIGURES

		Page
Figure 1	Substrate with Electrical Properties Varying in one direction	5
Figure 2	Surface with Simple Discontinuity in Electrical Properties	14
Figure 3	Illustration for Equations (43)	16
Figure 4	Illustration for Equation (49)	18
Figure 5	Illustration for Equations (58) - (63)	23
Figure 6	Variables for Stationary Phase Integrations of (121)	47
Figure 7	Surface with two discontinuities	55
Figure 8	Variables in Equation (165)	63
Figure 9	Surface Discontinuity in a Bounded Region R	65
Figure 10	Illustration for (184)	72
Figure 11	Illustration for Equation (196)	78
Figure 12	Illustration for Equation (205)	81
Figure 13	Illustration for (A1) and (A2)	96

1. INTRODUCTION

The use of high frequency (HF) radars as remote sensing devices for sea ice is being studied at Memorial University. This research project is a joint undertaking by the author and his students in the Faculty of Engineering, and Memorial's Centre for Cold Ocean Resources Engineering (C-CORE). A major iceberg detection experiment using HF radar will be conducted by the group in the Strait of Belle Isle in June 1983. This iceberg detection experiment forms a portion of a larger experiment being conducted at the same time by C-CORE. The other portion of the experiment is designed to provide an ocean surface current map for a portion of the Strait.

This report describes certain analytic work which began in 1982 while the author was on sabbatical leave with C-CORE. It is the third in a series of technical reports by the author on electromagnetic propagation and scatter relating to both the open ocean and an ice-infested ocean (Walsh, 1980, 1982). The ultimate aim of this work is to provide modelling techniques for propagation predictions, and to estimate radar cross sections for remote sensing of ocean surface parameters and sea ice. Also significant in this respect is the work of the author's students, Donnelly (1983), Ryan (1983a, 1983b) and Srivastava (1983a, 1983b). Further work underway and planned is discussed in the conclusions.

The analytic work reported here provides solutions for electromagnetic propagation and scatter for a surface whose electrical properties have one or more discontinuities. The solutions should prove useful for modelling propagation and scatter for water to ice or

ice to water interfaces, as well as the case of water to ice to water. A solution is also provided for the case of a finite, irregular discontinuity. The latter we feel may prove adequate for calculation of radar cross-sections for iceberg targets. A computer program is provided to make this calculation.

The analytic approach used in this report is the same as that already employed for the treatment of rough surface propagation and scatter (Walsh, 1980) and layered media propagation and scatter (Walsh, 1982), this approach differs considerably from previously published methods. The technique yields equations relating the unknown electrical field intensities and the known source fields directly without the use of Hertz vectors. Furthermore, the technique makes no a priori assumptions on the excitation or source, except that it is a physically realizable antenna system. Having said this, it should be mentioned that extensive use has been made of the vertical electric dipole as a source for two reasons; simplicity, and comparison with existing work (Bremmer, 1954, Wait, 1962, 1974). Also since HF radars will rely on surface wave propagation for over the horizon range they will, of necessity, use vertically polarized antenna systems.

Chapter 2 contains a general formulation of the problem of determining the electromagnetic field resulting from a finite electric current source located above a plane surface. The plane surface is assumed to have electrical properties subject to variation in one direction. The particular case when the variation is a simple discontinuity is treated in some detail. The formulation begins directly with Maxwell's equations.

Chapter 3 contains solutions to the problem as posed in Chapter 2 for the following cases: 1) a surface with one simple discontinuity, 2) a surface with two simple discontinuities and 3) a surface with a discontinuity involving a finite region. Expressions are given for the electric field in all regions of the surface in each case. These expressions are used in Chapter 4 to develop radar range equations and radar cross sections for surface wave radars operating in the monostatic mode.

Chapter 5, the Conclusions, contains two sections: a discussion in which the theories developed in this report are re-examined from the point of view of discussing some of the approximations made in writing Chapter 3 as well as plans for future work; and a section detailing a proposed model for the forthcoming iceberg detection experiment.

The Appendix contains a computer program for calculating the returned power and radar cross section for case 3 in Chapter 3. Several sample calcuations are given.

2. PROBLEM FORMULATION

2.1 Introduction

We shall, in this chapter, present a formulation for the problem of determining the electromagnetic field resulting from a finite electric source current located above a plane surface. The material bounded by the plane surface will be assumed to have electrical properties which vary in one direction. The situation is illustrated in Figure 1.

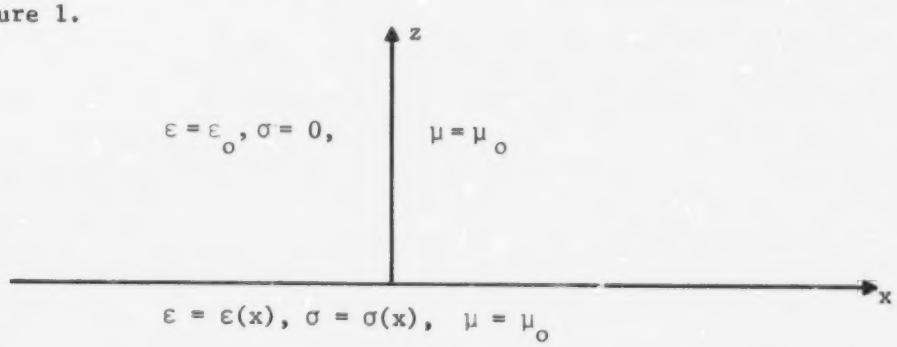


Figure 1 - Substrate with Electrical Properties Varying in one direction

Referring to Figure 1, we have,

$$\varepsilon = \begin{cases} \varepsilon_o & 0 < z < \infty \\ \varepsilon(x) & -\infty < z \leq 0 \end{cases}$$

$$\sigma = \begin{cases} 0 & 0 < z < \infty \\ \sigma(x) & -\infty < z \leq 0 \end{cases}$$

$$\mu = \mu_o, \quad -\infty < z < \infty$$

where

ε = the permittivity

μ = the permeability

σ = the electrical conductivity

ε_o = the permittivity of vacuum

μ_o = the permeability of vacuum.

We shall find it convenient to employ the Heaviside function:

$$h(z) = h_z = \begin{cases} 0, & z < 0 \\ 1, & z > 0 \end{cases}$$

Using this function, the electrical properties of the complete space may be summarized as,

$$\varepsilon = h_z \, \varepsilon_o + (1 - h_z) \, \varepsilon(x) \tag{1}$$

$$\sigma = (1 - h_z) \, \sigma(x) \tag{2}$$

The procedure which we shall follow is basically the same as that employed for rough surfaces (Walsh, 1980), and in the analysis of layered media propagation (Walsh, 1982).

2.2 Maxwell's Equations for the Complete Space

We now write Maxwell's equations in their point form, which we assume apply to the complete space:

$$\nabla \times \vec{E} = -j \, \omega \, \vec{B} \tag{3}$$

$$\nabla \times \vec{H} = j \, \omega \, \vec{D} + \vec{J} \tag{4}$$

$$\nabla \cdot \vec{B} = 0 \tag{5}$$

$$\nabla \cdot \vec{D} = \rho \tag{6}$$

where the field components have their usual notation. This is the time harmonic form or the Fourier transform with respect to the time variable of the time dependent equations. The transform variable is ω and $j = \sqrt{-1}$.

We shall assume point isotrophy, i.e.,

$$\vec{B} = \mu_o \vec{H} \quad \text{for all } z \tag{7}$$

$$\vec{D} = \begin{cases} \varepsilon_o \vec{E} , & z > 0 \\ \varepsilon(x)\vec{E}, & z < 0 \end{cases} \tag{8}$$

$$\vec{J}_c = \begin{cases} 0 , & z > 0 \\ \sigma(x)\vec{E}, & z < 0, \end{cases} \tag{9}$$

where $\vec{J}_c$ denotes the conduction currents. Equations (8) and (9) may be summarized as,

$$\vec{D} = [h_z \, \varepsilon_o + (1 - h_z) \, \varepsilon(x)] \, \vec{E} \tag{10}$$

$$\vec{J}_c = (1 - h_z) \, \sigma(x) \, \vec{E} \, . \tag{11}$$

We shall imterpret all of the above equations in the sense of generalized functions (Gel'fand and Shilov, 1964). It will be obvious, however, that our method of dealing with these equations implies certain restrictions on the behaviour of the fields and their spatial derivatives. The implications of these restrictions are that we are seeking solutions in a limited class of generalized functions and not in the widest sense.

2.3 The Basic Equation for the Electric Field

Consider (4), which we rewrite in the following form;

$$\nabla \times \vec{H} = j \omega \vec{D} + \vec{J}_c + \vec{J}_s = j \omega \left[\vec{D} + \frac{1}{j\omega} \vec{J}_c\right] + \vec{J}_s \tag{12}$$

where $\vec{J}_s$ is a source electric current with support in the half-space $z > 0$. The quantity in brackets on the right side of (12) we denote

as $\vec{D}_c$, and note from (10) and (11) that,

$$\vec{D}_c = \vec{D} + \frac{1}{j\omega} \vec{J}_c$$

$$= [h_z \varepsilon_o + (1 - h_z)(\varepsilon(x) - j \frac{\sigma(x)}{\omega})] \vec{E} \qquad (13)$$

$$= \varepsilon_o [h_z + (1 - h_z) n_0^2(x)] \vec{E}$$

where

$$n_0^2(x) = \frac{\varepsilon(x)}{\varepsilon_o} - j \frac{\sigma(x)}{\varepsilon_o \omega} = \varepsilon_r(x) - j \frac{\sigma(x)}{\varepsilon_o \omega}$$

is the square of the refractive index. We note that (13) may be inverted to yield,

$$\vec{E} = \frac{1}{\varepsilon_o} [h_z + (1 - h_z)(n_0^2)^{-1}] \vec{D}_c. \qquad (14)$$

We now proceed by taking the curl of (3), and applying (12)

$$\nabla \times \nabla \times \vec{E} = - j \omega \mu_o \nabla \times \vec{H}$$

$$= - j \omega \mu_o [j \omega \vec{D}_c + \vec{J}_s]$$

$$= k^2 [h_z + (1 - h_z) n_0^2(x)] \vec{E} - j \omega \mu_o \vec{J}_s \qquad (15)$$

or,

$$\nabla \times \nabla \times \vec{E} - k^2 [h_z + (1 - h_z) n_0^2(x)] \vec{E} = -j \omega \mu_0 \vec{J}_s,$$

where $k^2 = \omega^2 \mu_o \varepsilon_o$ is the square of the freespace wavenumber.

If we assume that $\vec{E}$ satisfies

$$\nabla \times \nabla \times \vec{E} = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E}, \qquad (16)$$

then (15) takes the form,

$$\nabla^2 \vec{E} + k^2 [h_z + (1 - h_z) n_0^2(x)] \vec{E} = \nabla (\nabla.\vec{E}) + j \omega \mu_o \vec{J}_s \qquad (17)$$

and we need to interpret $\nabla(\nabla \cdot \vec{E})$.

Refer to (14) and note we may write that equation in the form

$$\vec{E} = \frac{1}{\varepsilon_o} [(n_0^2)^{-1} + h_z (1 - (n_0^2)^{-1})] \vec{D}_c, \tag{18}$$

or,

$$\vec{E} = \frac{1}{\varepsilon_o} [1 + (1 - h_z)(n_0^2)^{-1} - 1)] \vec{D}_c. \tag{19}$$

Let us first consider (18). We have,

$$\nabla \cdot \vec{E} = \frac{1}{\varepsilon_o} [(n_0^2)^{-1} + h_z (1 - (n_0^2)^{-1}] \nabla \cdot \vec{D}_c$$

$$+ \frac{1}{\varepsilon_o} \frac{\partial (n_0^2)^{-1}}{\partial x} (1 - h_z) D_{cx} + \frac{1}{\varepsilon_o} (1 - (n_0^2)^{-1}) \delta_z D_{cz}^{z+} \tag{20}$$

where $\delta_z = \delta(z)$ = Dirac delta function and D_{cx} and D_{cz} are the x and z components of $\vec{D}_c$ respectively. Also,

$$D_{cz}^{z+} = \lim_{z \to o^+} D_{cz}, \tag{21}$$

i.e., the value of D_{cz} immediately above the interface.

Refer now to (12). If we assume that the fields satisfy the usual vector identities, and also recall that the support of $\vec{J}_s$ is the open half-space $z > 0$, i.e., $\vec{J}_s = h_z \vec{J}_s$, we have

$$\nabla \cdot \nabla \times \vec{H} \equiv 0 = j \omega \nabla \cdot \vec{D}_c + \nabla \cdot \vec{J}_s, \tag{22}$$

or

$$\nabla \cdot \vec{D}_c = - \frac{1}{j\omega} \nabla \cdot \vec{J}_s \quad .$$

In particular,

$$\nabla \cdot \vec{D}_c = 0, \quad z < 0.$$

Also, from (13)

$$(1 - h_z)\, D_{cx} = \varepsilon_o\, n_0^2\, (1 - h_z)\, E_x \tag{23}$$

$$D_{cz}^{z+} = \varepsilon_o\, E_z^{z+} \quad .$$

Using then (22) and (23), we see that (20) may be written as,

$$\nabla \cdot \vec{E} = -\frac{1}{j\,\omega\,\varepsilon_o}\,\nabla \cdot \vec{J}_s + n_0^2\,\frac{\partial (n_0^2)^{-1}}{\partial x}\,(1 - h_z)\,E_x$$

$$+ (1 - (n_0^2)^{-1})\,\delta_z\,E_z^{z+} \quad . \tag{24}$$

A second expression for $\nabla \cdot \vec{E}$ may be obtained from (19). We have

$$\nabla \cdot \vec{E} = -\frac{1}{j\,\omega\,\varepsilon_o}\,\nabla \cdot \vec{J}_s + n_0^2\,\frac{\partial (n_0^2)^{-1}}{\partial x}\,(1 - h_z)\,E_x$$

$$+ \frac{1}{\varepsilon_o}\,(1 - (n_0^2)^{-1})\,\delta_z\,D_{cz}^{z-} \tag{25}$$

where we have used (22) and (23) and

$$D_{cz}^{z-} = \lim_{z \to 0-} D_{cz},$$

i.e., the value of D_{cz} immediately below the interface. However, from (13),

$$D_{cz}^{z-} = \varepsilon_o\, n_0^2\, E_z^{z-} \quad .$$

Therefore (25) may be written as

$$\nabla \cdot \vec{E} = -\frac{1}{j\,\omega\,\varepsilon_o}\,\nabla \cdot \vec{J}_s + n_0^2\,\frac{\partial (n_0^2)^{-1}}{\partial x}\,(1 - h_z)\,E_x$$

$$+ n_0^2\,(1 - (n_0^2)^{-1})\,\delta_z\,E_z^{z-}. \tag{26}$$

Comparing (24) and (26) implies, since $\nabla \cdot \vec{E}$ should be unique, that

$$E_z^{z+} = n_0^2 \, E_z^{z-}, \tag{27}$$

or, a discontinuity in the vertical component of the electric field at the interface. This, of course, agrees with established theory.

We can now compute $\nabla(\nabla \cdot \vec{E})$. Referring to (24) we have,

$$\nabla(\nabla \cdot \vec{E}) = - \frac{1}{j \omega \varepsilon_o} \nabla(\nabla \cdot \vec{J}_s) + \nabla[n_0^2 \frac{\partial(n_0^2)^{-1}}{\partial x} (1 - h_z) E_x]$$

$$+ \nabla[1 - (n_0^2)^{-1}) \delta_z E_z^{z+}].$$

The second term on the right hand side of the above may be rewritten as follows,

$$\nabla[n_0^2 \frac{\partial(n_0^2)^{-1}}{\partial x} (1 - h_z) E_x] = (1 - h_z) \nabla[n_0^2 \frac{\partial(n_0^2)^{-1}}{\partial x} E_x]$$

$$- n_0^2 \frac{\partial(n_0^2)^{-1}}{\partial x} E_x^{z-} \delta_z \vec{a}_z,$$

where

$$E_x^{z-} = \lim_{z \to 0-} E_x,$$

i.e., E_x immediately below the interface, and $\vec{a}_z$ is the unit vector in the z-directon. Also, the last term is

$$\nabla[(1 - (n_0^2)^{-1} \delta_z E_z^{z+}] = \nabla_{xy} [(1 - (n_0^2)^{-1}) E_z^{z+}] \delta_z$$

$$+ [(1 - (n_0^2)^{-1} E_z^{z+}] \delta_z' \vec{a}_z,$$

where ∇_{xy} is the restriction of the gradient to x and y only and δ_z' is the z derivative of the Dirac delta function $\delta_z = \delta(z)$. Therefore,

$$\nabla(\nabla \cdot \vec{E}) = -\frac{1}{j\omega\varepsilon_o}\nabla(\nabla \cdot \vec{J}_s) + (1 - h_z)\,\nabla\left[n_0^2 \frac{\partial(n_0^2)^{-1}}{\partial x} E_x\right]$$

$$+ \nabla_{xy}\left[(1 - (n_0^2)^{-1})\, E_z^{z+}\right] \delta_z$$

$$+ \left[(1 - (n_0^2)^{-1})\, E_z^{z+}\, \delta_z' - n_0^2 \frac{\partial(n_0^2)^{-1}}{\partial x} E_x^{z-}\, \delta_z\right] \vec{a}_z, \qquad (28)$$

and equation (17) may be written as,

$$\nabla^2 \vec{E} + k^2[h_z + (1 - h_z)\, n_0^2]\, \vec{E} - (1 - h_z)\,\nabla\left[n_0^2 \frac{\partial(n_0^2)^{-1}}{\partial x} E_x\right]$$

$$= -T_{SE}(\vec{J}_s) + \nabla_{xy}\left[(1 - (n_0^2)^{-1})\, E_z^{z+}\right] \delta_z$$

$$+ \left[(1 - (n_0^2)^{-1})\, E_z^{z+}\, \delta_z' - n_o^2 \frac{\partial(n_0^2)^{-1}}{\partial x} E_x^{z-}\, \delta_z\right] \vec{a}_z, \qquad (29)$$

where

$$T_{SE} = \frac{1}{j\omega\varepsilon_o}[\nabla(\nabla\ \cdot\ ) + k^2] \qquad (30)$$

is the electrical source operator.

Equation (29), is then the basic differential equation that the expression for the electric field $\vec{E}$ must satisfy. A similar equation may be derived in terms of the magnetic field $\vec{H}$. In the next section we shall reduce (29) to a form which will be used exclusively in the sequel.

2.4 The Field Decomposition

Let us write the expression for $\vec{E}$ in the form

$$\vec{E} = h_z \vec{E} + (1 - h_z)\vec{E} \qquad (31)$$

Then

$$\nabla^2 \vec{E} = \nabla^2(h_z \vec{E}) + \nabla^2 [(1 - h_z) \vec{E}]$$

$$= h_z \nabla^2 \vec{E} + (1 - h_z) \nabla^2 \vec{E}$$

$$+ (\vec{E}^{z+} - \vec{E}^{z-}) \delta'_z + [(\frac{\partial \vec{E}}{\partial z})^{z+} - (\frac{\partial \vec{E}}{\partial z})^{z-}] \delta_z, \tag{32}$$

where the superscripts have the same definition as in (29). Substituting (32) into (29) we see that a solution of (29) is given by $\vec{E}$ which satisfies,

$$h_z[\nabla^2 \vec{E} + k^2 \vec{E}] = - T_{SE} (\vec{J}_s) \tag{33}$$

$$(1 - h_z) [\nabla^2 \vec{E} + k^2 n_0^2 \vec{E} - \nabla(n_0^2 \frac{\partial (n_0^2)^{-1}}{\partial x} E_x)] = 0 \tag{34}$$

$$\vec{E}^{z+} - \vec{E}^{z-} = (1 - (n_0^2)^{-1}) E_z^{z+} \hat{a}_z \tag{35}$$

$$(\frac{\partial \vec{E}}{\partial z})^{z+} - (\frac{\partial \vec{E}}{\partial z})^{z-} = \nabla_{xy} [(1 - (n_0^2)^{-1}) E_z^{z+}] - n_0^2 \frac{\partial (n_0^2)^{-1}}{\partial x} E_x^{z-} \hat{a}_z \quad . \tag{36}$$

It should be noted that (33) and (34) are differential equation which the expression for $\vec{E}$ must satisfy on the interior of the half-spaces determined by h_z and $(1-h_z)$. Equations (35) and (36) are boundary equations which we rewrite as,

$$\vec{E}^{z-} = \vec{E}^{z+} - (1 - (n_0^2)^{-1}) E_z^{z+} \hat{a}_z \tag{37}$$

$$(\frac{\partial \vec{E}}{\partial z})^{z-} = (\frac{\partial \vec{E}}{\partial z})^{z+} - \nabla_{xy} [(1 - (n_0^2)^{-1}) E_z^{z+}] \tag{38}$$

$$+ n_0^2 \frac{\partial (n_0^2)^{-1}}{\partial x} E_x^{z+} \hat{a}_z,$$

i.e., the quantities with negative superscripts in terms of those with positive superscripts. Also note that we have replaced E_x^{z-} in (36) with E_x^{z+} in (38), since (35) and (37) imply $E_x^{z+} = E_x^{z-}$.

2.5 Reduction to Integral Equations in the Case of a Simple Discontinuity

In this section the problem as represented by equations (33) - (38) will be reduced to one involving integral equations of the convolution type. In order to accomplish this we will assume a particular form for $n_0^2(x)$, i.e., the refractive index in the lower media.

Refer now to equation (13). We write

$$n_0^2(x) = \varepsilon_r(x) - j\frac{\sigma(x)}{\varepsilon_o \omega}$$

$$= [\varepsilon_{r1} h(x) + \varepsilon_{r2}(1 - h(x))] - j\left[\frac{\sigma_1}{\varepsilon_o \omega} h(x) + \frac{\sigma_2}{\varepsilon_o \omega}(1 - h(x))\right]$$

$$= n_{01}^2 h_x + n_{02}^2 (1 - h_x) \qquad (39)$$

where

$$h(x) = h_x = \begin{cases} 0, & x < 0 \\ 1, & x > 0. \end{cases}$$

The situation is illustrated in Figure 2, i.e., the variation of the electrical properties of the lower media consists of a simple discontinuity at the origin.

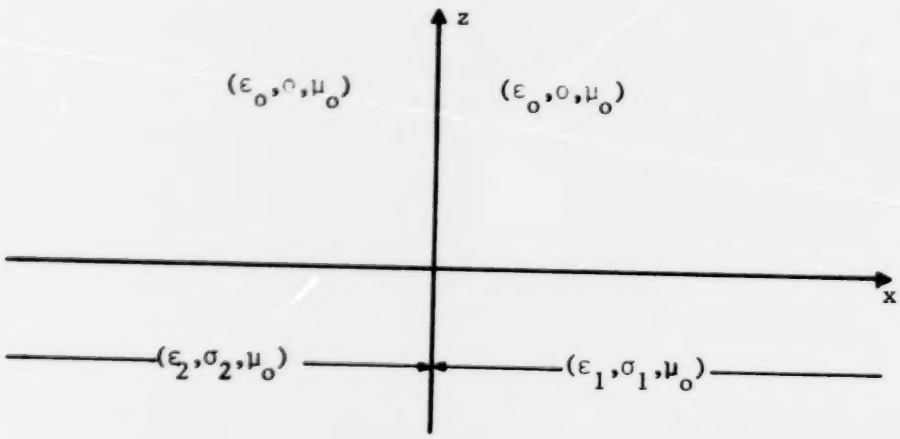


Figure 2 Surface with Simple Discontinuity in Electrical Properties

Let us now introduce a second decomposition of $\vec{E}$. Let

$$\vec{E} = h_x \vec{E} + (1 - h_x)\, \vec{E}, \tag{40}$$

then,

$$\nabla^2 \vec{E} = h_x \nabla^2 \vec{E} + (1 - h_x)\, \nabla^2 \vec{E} + (\vec{E}^{x+} - \vec{E}^{x-})\, \delta'_x$$

$$+ [(\frac{\partial \vec{E}}{\partial x})^{x+} - (\frac{\partial \vec{E}}{\partial x})^{x-}] \delta_x \tag{41}$$

where $\delta_x = \delta(x)$ is the Dirac delta function with argument x and δ'_x its derivative, and

$$\vec{E}^{x+} = \lim_{x \to 0^+} \vec{E}$$

$$\vec{E}^{x-} = \lim_{x \to 0^-} \vec{E}$$

$$(\frac{\partial \vec{E}}{\partial x})^{x+} = \lim_{x \to 0+} \frac{\partial \vec{E}}{\partial x}$$

$$(\frac{\partial \vec{E}}{\partial x})^{x-} = \lim_{x \to 0^-} \frac{\partial \vec{E}}{\partial x}\,.$$

First consider (33), i.e.,

$$h_z\, [\nabla^2 \vec{E} + k^2 \vec{E}] = - T_{SE}\, (\vec{J}_s).$$

Using (39) and (40) and (41) we obtain,

$$h_z\, [h_x\, (\nabla^2 \vec{E} + k^2 \vec{E}) + (1 - h_x)(\nabla^2 \vec{E} + k^2 \vec{E})$$

$$+ (\vec{E}^{x+} - \vec{E}^{x-})\, \delta_x^- + \{(\frac{\partial \vec{E}}{\partial x})^{x+} - (\frac{\partial \vec{E}}{\partial x})^{x-}\}\, \delta_x] = - T_{SE}\, (\vec{J}_s) \tag{42}$$

Obviously, a solution of (42) is given by $\vec{E}$ which satisfies,

$$h_z\, h_x\, (\nabla^2 \vec{E} + k^2 \vec{E}) = 0 \tag{43a}$$

$$h_z\, (1 - h_x)(\nabla^2 \vec{E} + k^2 \vec{E}) = - T_{SE}\, (\vec{J}_s) \tag{43b}$$

$$h_z \vec{E}^{x-} = h_z \vec{E}^{x+} \tag{43c}$$

$$h_z \left(\frac{\partial \vec{E}}{\partial x}\right)^{x-} = h_z \left(\frac{\partial \vec{E}}{\partial x}\right)^{x+}, \tag{43d}$$

where we have assumed that the support of $\vec{J}_s$ lies in the 2nd quadrant, i.e., $\vec{J}_s = h_z (1 - h_x) \vec{J}_s$.

Equations (43a) and (43b) are partial differential equations that the expression for $\vec{E}$ must satisfy in the first and second quadrants respectively, while (43c) and (43d) simply assert the continuity of $\vec{E}$ and the normal derivative $\partial\vec{E}/\partial x$ at the boundary of these two regions.

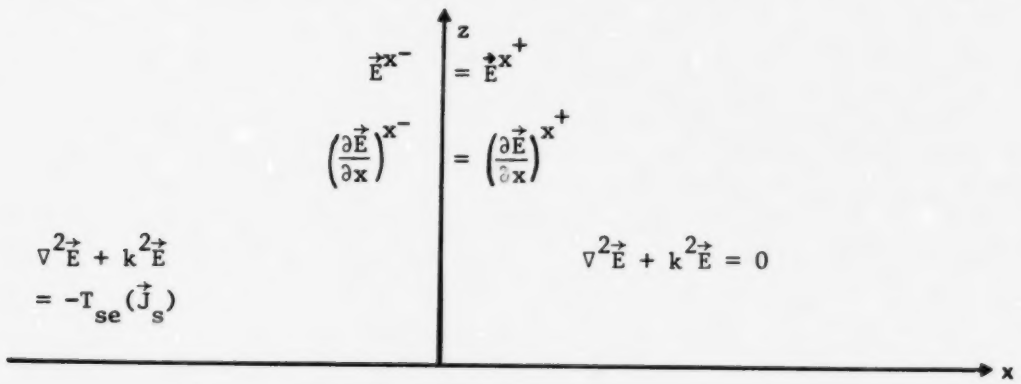


Figure 3 - Illustration for Equations (43)

Refer now to (34), i.e.,

$$(1 - h_z)[\nabla^2 \vec{E} + k^2 n_0^2 \vec{E} - \nabla \left(n_0^2 \frac{\partial (n_0^2)^{-1}}{\partial x} E_x\right)] = 0.$$

Now from (39),

$$n_0^2 = n_{01}^2 h_x + n_{02}^2 (1 - h_x),$$

hence,

$$(n_0^2)^{-1} = \frac{1}{n_{01}^2} h_x + \frac{1}{n_{02}^2} (1 - h_x). \tag{44}$$

Then

$$n_o^2 \frac{\partial (n_o^2)^{-1}}{\partial x} E_x = \frac{\partial}{\partial x}\left[\frac{1}{n_{02}^2} + \frac{n_{02}^2 - n_{01}^2}{n_{02}^2\, n_{01}^2} h_x\right]\left[n_{01}^2 h_x + n_{02}^2 (1 - h_x)\right] E_x$$

$$= \frac{n_{02}^2 - n_{01}^2}{n_{02}^2} E_x^{x+} \delta_x . \tag{45}$$

Also,

$$n_0^2 \frac{\partial (n_0^2)^{-1}}{\partial x} E_x = \frac{\partial}{\partial x}\left[\frac{1}{n_{01}^2} - \frac{n_{02}^2 - n_{01}^2}{n_{01}^2\, n_{02}^2}(1 - h_x)\right]\left[n_{01}^2 h_x + n_{02}^2 (1 - h_x)\right].$$

$$.E_x = \frac{n_{02}^2 - n_{01}^2}{n_{01}^2} E_x^{x-} \delta_x . \tag{46}$$

Comparing (45) and (46) we see that for uniqueness,

$$\frac{E_x^{x-}}{n_{01}^2} = \frac{E_x^{x+}}{n_{02}^2} . \tag{47}$$

Using then (39), (40), (41) and (45), (34) may be written as,

$$(1 - h_z)\left[h_x (\nabla^2 \vec{E} + n_{01}^2 k^2 \vec{E}) + (1 - h_x)(\nabla^2 \vec{E} + n_{02}^2 k^2 \vec{E})\right.$$

$$\left. + (\vec{E}^{x+} - \vec{E}^{x-})\, \delta_x' + \left[\left(\frac{\partial \vec{E}}{\partial x}\right)^{x+} - \left(\frac{\partial \vec{E}}{\partial x}\right)^{x-}\right] \delta_x\right]$$

$$= (1 - h_z)\, \nabla \left[\frac{n_{02}^2 - n_{01}^2}{n_{02}^2} E_x^{x+} \delta_x\right]$$

$$= (1 - h_z)\left[\frac{n_{02}^2 - n_{01}^2}{n_{02}^2} \{\nabla_{yz} (E_x^{x+})\, \delta_x + E_x^{x+} \delta_x' \vec{a}_x\}\right].$$

(48)

A solution of (48) is given by an expression for $\vec{E}$ which satisfies,

$$(1 - h_z)\, h_x\, (\nabla^2 \vec{E} + n_{01}^2\, k^2\, \vec{E}) = 0 \tag{49a}$$

$$(1 - h_z)(1 - h_x)(\nabla^2 \vec{E} + n_{02}^2\, k^2\, \vec{E}) = 0 \tag{49b}$$

$$(1 - h_z)(\vec{E}^{x+} - \vec{E}^{x-}) = (1 - h_z)\, \frac{n_{02}^2 - n_{01}^2}{n_{01}^2}\, E_x^{x+}\, \vec{a}_x \tag{49c}$$

$$(1 - h_z)[(\frac{\partial \vec{E}}{\partial x})^{x+} - (\frac{\partial \vec{E}}{\partial x})^{x-}] = (1 - h_z)\, \frac{n_{02}^2 - n_{01}^2}{n_{02}^2}\, \nabla_{yz}\, (E_x^{x+}) \quad . \tag{49d}$$

Again, as in (43), (49a) and (49b) are partial differential equations that the expression for $\vec{E}$ must satisfy in the fourth and third quadrants respectively, while (49c) and (49d) are the relationships that the electric field and its normal derivative must satisfy at the interface of these two regions.

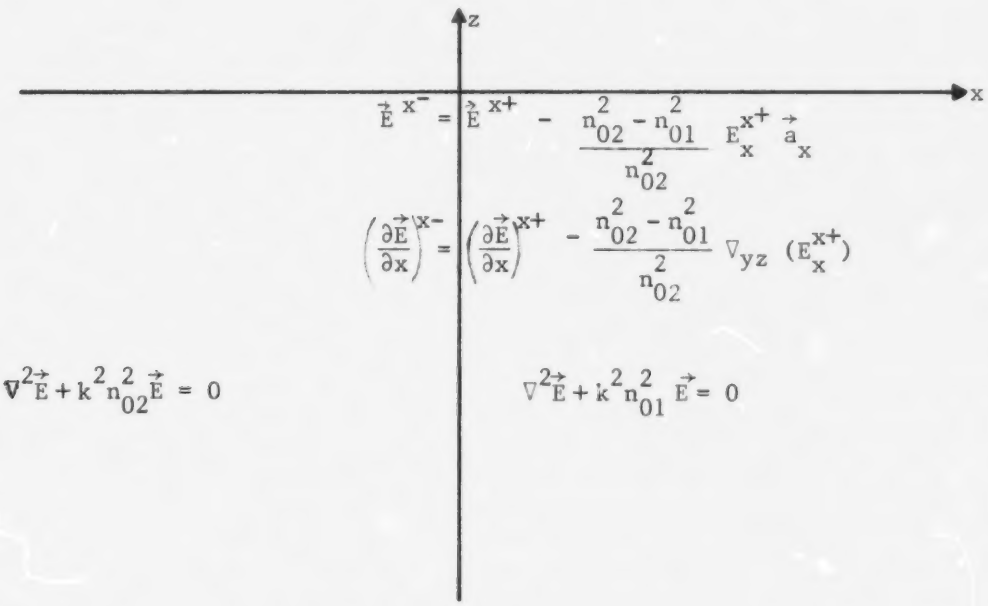

z
x
$\vec{E}^{x-} = \vec{E}^{x+} - \frac{n_{02}^2 - n_{01}^2}{n_{02}^2} E_x^{x+} \vec{a}_x$
$\left(\frac{\partial \vec{E}}{\partial x}\right)^{x-} = \left(\frac{\partial \vec{E}}{\partial x}\right)^{x+} - \frac{n_{02}^2 - n_{01}^2}{n_{02}^2} \nabla_{yz} (E_x^{x+})$
$\nabla^2\vec{E} + k^2 n_{02}^2 \vec{E} = 0$
$\nabla^2\vec{E} + k^2 n_{01}^2 \vec{E} = 0$

Figure 4 Illustration for Equation (49)

Refer now to (37) and (38), the boundary equations for the xy plane. First consider (37), i.e.,

$$\vec{E}^{z-} = \vec{E}^{z+} - (1 - (n_{02}^2)^{-1})\, E_z^{z+}\, \vec{a}_z .$$

Using (44) and (40), this equation may be rewritten as,

$$h_x \vec{E}^{z-} + (1 - h_x)\, \vec{E}^{z-} = h_x \vec{E}^{z+} + (1 - h_x)\, \vec{E}^{z+}$$

$$- [1 - (\frac{1}{2}_{n_{01}} h_x + \frac{1}{2}_{n_{02}} (1 - h_x)][h_x\, E_z^{z+} + (1 - h_x)\, E_z^{z+}]\, \vec{a}_z .$$

Comparing terms with like support gives,

$$h_x \vec{E}^{z-} = h_x (\vec{E}^{z+} - \frac{n_{01}^2 - 1}{n_{01}^2}\, E_z^{z+}\, \vec{a}_z) \qquad (50a)$$

$$(1 - h_x)\, \vec{E}^{z-} = (1 - h_x)(\vec{E}^{z+} - \frac{n_{02}^2 - 1}{n_{02}^2}\, E_z^{z+}\, \vec{a}_z) \quad . \qquad (50b)$$

Consider now (38), i.e,

$$(\frac{\partial \vec{E}}{\partial z})^{z-} = (\frac{\partial \vec{E}}{\partial z})^{z+} - \nabla_{xy}\, [(1 - (n_0^2)^{-1})\, E_z^{z+}] + n_0^2\, \frac{\partial (n_0^2)^{-1}}{\partial x}\, E_z^{z+}\, \vec{a}_z \quad .$$

From (44)

$$1 - (n_0^2)^{-1} = 1 - \frac{1}{n_{01}^2} h_x - \frac{1}{n_{02}^2} (1 - h_x)$$

$$= \frac{n_{01}^2 - 1}{n_{01}^2} h_x + \frac{n_{02}^2 - 1}{n_{02}^2} (1 - h_x). \qquad (51)$$

Equation (51) may be written in two ways, i.e.,

$$1 - (n_0^2)^{-1} = \frac{n_{02}^2 - 1}{n_{02}^2} + \left[\frac{1}{n_{02}^2} - \frac{1}{n_{01}^2}\right] h_x \tag{52a}$$

$$1 - (n_0^2)^{-1} = \frac{n_{01}^2 - 1}{n_{01}^2} - \left[\frac{1}{n_{02}^2} - \frac{1}{n_{01}^2}\right] (1 - h_x). \tag{52b}$$

Therefore,

$$\frac{\partial}{\partial x}\left[(1 - (n_0^2)^{-1} E_z^{z+}\right] = \frac{n_{01}^2 - 1}{n_{01}^2} h_x \frac{\partial E_z^{z+}}{\partial x} + \frac{n_{02}^2 - 1}{n_{02}^2} (1 - h_x) \frac{\partial E_z^{z+}}{\partial x}$$

$$+ \left[\frac{1}{n_{02}^2} - \frac{1}{n_{01}^2}\right] (E_z^{z+})^{x+} \delta_x \tag{53a}$$

or

$$= \frac{n_{01}^2 - 1}{n_{01}^2} h_x \frac{\partial E_z^{z+}}{\partial x} + \frac{n_{02}^2 - 1}{n_{02}^2} (1 - h_x) \frac{\partial E_z^{z+}}{\partial x}$$

$$+ \left[\frac{1}{n_{02}^2} - \frac{1}{n_{01}^2}\right] (E_z^{z+})^{x-} \delta_x, \tag{53a}$$

where

$$(E_z^{z+})^{x+} = \lim_{x \to o^+} E_z^{z+}$$

$$(E_z^{z+})^{x-} = \lim_{x \to o^-} E_z^{z+} \quad .$$

But (43c) requires,

$$(E_z^{z+})^{x+} = (E_z^{z+})^{x-}.$$

So for uniqueness, we must have,

$$(E_z^{z+})^{x+} = (E_z^{z+})^{x-} = 0, \tag{54}$$

$n_{01}^2 \neq n_{02}^2$. We therefore write,

$$\nabla_{xy}\left[(1-(n_0^2)^{-1}\, E_z^{z+}\right] = \frac{n_{01}^2 - 1}{n_{01}^2}\, h_x\, \nabla_{xy}\,(E_z^{z+}) + \frac{n_{02}^2 - 1}{n_{02}^2}\,(1 - h_x)\,[\nabla_{xy}\,(E_z^{z+})]. \tag{55}$$

By a similar argument and an appeal to (43c) we have

$$(E_x^{z+})^{x-} = (E_x^{z+})^{x+} = 0, \tag{56}$$

and hence

$$n_0^2\, \frac{\partial (n_0^2)^{-1}}{\partial x}\, E_x^{z+} = 0.$$

Therefore, (38) may be written as,

$$h_x\,(\frac{\partial \vec{E}}{\partial z})^{z-} = h_x\,[(\frac{\partial \vec{E}}{\partial z})^{z+} - \frac{n_{01}^2 - 1}{n_{01}^2}\,\nabla_{xy}\,(E_z^{z+})] \tag{57a}$$

$$(1 - h_x)\,(\frac{\partial \vec{E}}{\partial z})^{z-} = (1 - h_x)\,[(\frac{\partial \vec{E}}{\partial z})^{z+} - \frac{n_{02}^2 - 1}{n_{02}^2}\,\nabla_{xy}\,(E_z^{z+})]\,. \tag{57b}$$

Let us summarize now the relevent equations for each quadrant and the boundary conditions:

$$h_z\, h_x\,(\nabla^2 \vec{E} + k^2\, \vec{E}) = 0 \tag{58a}$$

$$h_z(1 - h_x)\,(\nabla^2 \vec{E} + k^2\, \vec{E}) = -T_{SE}\,(\vec{J}_S) \tag{58b}$$

$$(1 - h_z)(1 - h_x)\,(\nabla^2 \vec{E} + n_{02}^2\, k^2\, \vec{E}) = 0 \tag{58c}$$

$$(1 - h_z)\, h_x\,(\nabla^2 \vec{E} + n_{01}^2\, k^2\, \vec{E}) = 0 \tag{58d}$$

$$h_z \vec{E}^{x-} = h_z \vec{E}^{x+} \tag{59a}$$

$$h_z \left(\frac{\partial \vec{E}}{\partial x}\right)^{x-} = h_z \left(\frac{\partial \vec{E}}{\partial x}\right)^{x+} \tag{59b}$$

$$(1 - h_z)\vec{E}^{x-} = (1 - h_z)\left[\vec{E}^{x+} - \frac{n_{02}^2 - n_{01}^2}{n_{02}^2} E_x^{x+} \vec{a}_x\right] \tag{60a}$$

$$(1 - h_z)\left(\frac{\partial \vec{E}}{\partial x}\right)^{x-} = (1 - h_z)\left[\left(\frac{\partial \vec{E}}{\partial x}\right) - \frac{n_{02}^2 - n_{01}^2}{n_{02}^2} \nabla_{yz} (E_x^{x+})\right] \tag{60b}$$

$$h_x \vec{E}^{z-} = h_x \left[\vec{E}^{z+} - \frac{n_{01}^2 - 1}{n_{01}^2} E_z^{z+} \vec{a}_z\right] \tag{61a}$$

$$h_x \left(\frac{\partial \vec{E}}{\partial z}\right)^{z-} = h_x \left[\left(\frac{\partial \vec{E}}{\partial z}\right)^{z+} - \frac{n_{01}^2 - 1}{n_{01}^2} \nabla_{xy} (E_z^{z+})\right] \tag{61b}$$

$$(1 - h_x)\vec{E}^{z-} = (1 - h_x)\left[\vec{E}^{z+} - \frac{n_{02}^2 - 1}{n_{02}^2} E_z^{z+} \vec{a}_z\right] \tag{62a}$$

$$(1 - h_x)\left(\frac{\partial \vec{E}}{\partial z}\right)^{z-} = (1 - h_x)\left[\left(\frac{\partial \vec{E}}{\partial z}\right)^{z+} - \frac{n_{02}^2 - 1}{n_{02}^2} \nabla_{xy} (E_z^{z+})\right] \tag{62b}$$

$$(E_x^{z+})^{x+} = (E_x^{z+})^{x-} = (E_x^{z-})^{x+} = (E_x^{z-})^{x-} = 0 \tag{63a}$$

$$(E_z^{z+})^{x+} = (E_z^{z+})^{x-} = (E_z^{z-})^{x+} = (E_z^{z-})^{x-} = 0 \quad . \tag{63b}$$

Figure 5 shows the quantities in (58) - (63).

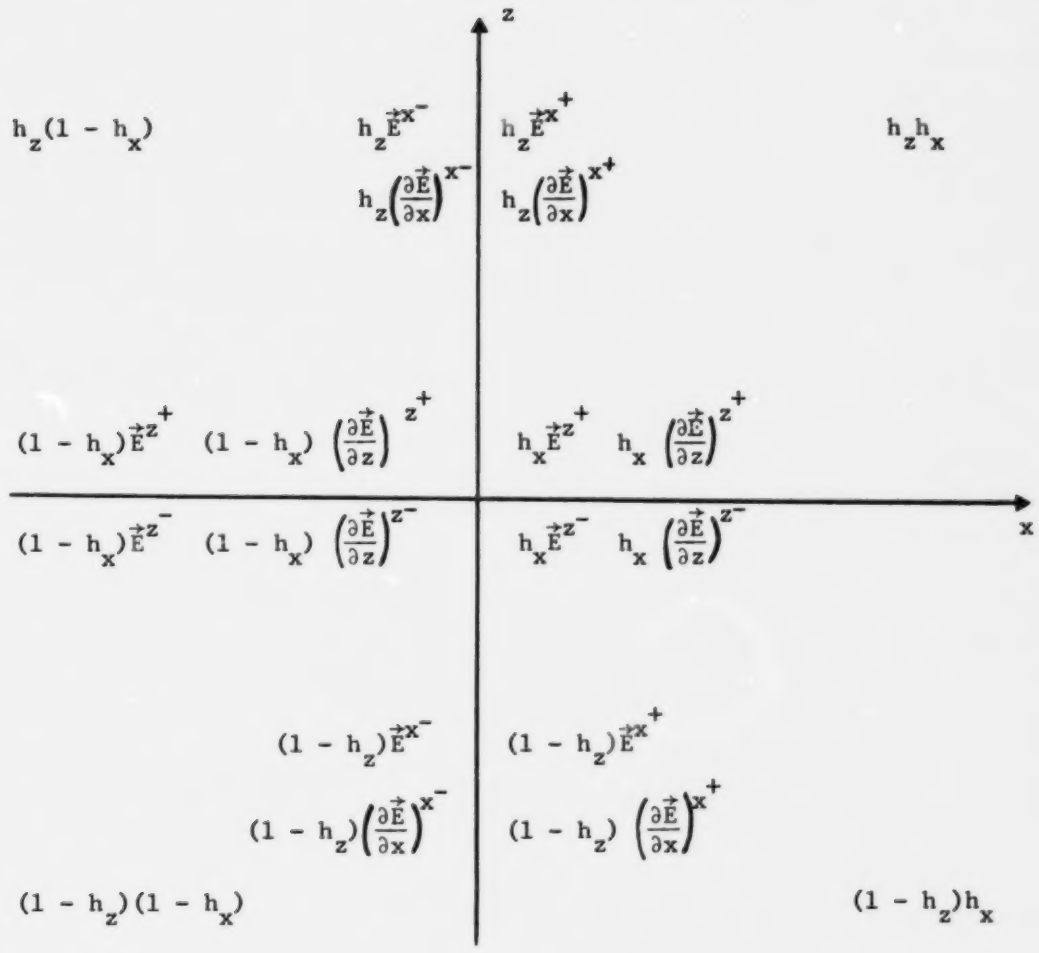


Figure 5 - Illustration for Equations (58) - (63).

We shall now reduce the problem of finding an expression for the electric field $\vec{E}$, satisfying (58)-(63) to an equivalent problem in the form of integral equations.

The partial differential equations

$$\nabla^2 K_0 + k^2 K_0 = -\delta(x)\ \delta(y)\ \delta(z)$$

$$\nabla^2 K_1 + k^2\ n_{01}^2\ K_1 = -\delta(x)\ \delta(y)\ \delta(z) \qquad (64)$$

$$\nabla^2 K_2 + k^2\ n_{02}^2\ K_2 = -\delta(x)\ \delta(y)\ \delta(z),$$

have the solutions (fundamental),

$$K_0(x,y,z) = \frac{e^{-jkr}}{4\pi r}$$

$$K_1(x,y,z) = \frac{e^{-jkn_{01}r}}{4\pi r} \qquad , \qquad r^2 = x^2+y^2+z^2 \tag{65}$$

$$K_2(x,y,z) = \frac{e^{-jkn_{02}r}}{4\pi r}$$

respectively. These solutions satisfy the Sommerfeld radiation condition. Consider the identity, valid in the sense of generalized functions,

$$\nabla^2(h_x h_z \vec{E}) * K_o \equiv h_x h_z \vec{E} * \nabla^2 K_o, \tag{66}$$

where * denotes 3-dimensional (spatial) convolution. From (64) the right side of (66)

$$= h_x h_z \vec{E} * [-\delta(x)\ \delta(y)\ \delta(z) - k^2 K_o]$$

$$= -h_x h_z \vec{E} - h_x h_z k^2 \vec{E} * K_o \tag{67}$$

while the left side

$$= [h_x h_z \nabla^2 \vec{E} + h_x \vec{E}^{z+} \delta'_z + h_x \left(\frac{\partial \vec{E}}{\partial z}\right)^{z+} \delta_z$$

$$+ h_z \vec{E}^{x+} \delta'_x + h_z \left(\frac{\partial \vec{E}}{\partial x}\right)^{x+} \delta_x] * K_o \ . \tag{68}$$

Equating (67) and (68) gives after rearranging,

$$h_x h_z \vec{E} = -h_x h_z (\nabla^2 \vec{E} + k^2 \vec{E}) * K_o - [h_x \{\vec{E}^{z+} \delta'_z + \left(\frac{\partial \vec{E}}{\partial z}\right)^{z+} \delta_z\}$$

$$+ h_z \{\vec{E}^{x+} \delta'_x + \left(\frac{\partial \vec{E}}{\partial x}\right)^{x+} \delta_x\}] * K_o \ . \tag{69}$$

Assuming $\vec{E}$ satisfies (58a), (69) becomes

$$h_x h_z \vec{E} = -[h_x\{\vec{E}^{z+}\,\delta_z' + (\frac{\partial \vec{E}}{\partial z})^{z+}\,\delta_z\} + h_z\{\vec{E}^{x+}\,\delta_x' + (\frac{\partial \vec{E}}{\partial x})^{x+}\,\delta_x\}] * K_o \,. \tag{70}$$

Similarly,

$$h_z(1-h_x)\vec{E} = [T_{SE}(\vec{J}_s) - (1-h_x)\{\vec{E}^{z+}\,\delta_z' + (\frac{\partial \vec{E}}{\partial z})^{z+}\,\delta_z\} + h_z\{\vec{E}^{x+}\,\delta_x' + (\frac{\partial \vec{E}}{\partial x})^{x-}\,\delta_x\}] * K_o$$

$$(1-h_z)(1-h_x)\vec{E} = [(1-h_x)\{\vec{E}^{z-}\,\delta_z' + (\frac{\partial \vec{E}}{\partial z})^{z-}\,\delta_z\} + (1-h_z)\{\vec{E}^{x-}\,\delta_x' + (\frac{\partial \vec{E}}{\partial x})^{x-}\,\delta_x\}] * K_2 \tag{72}$$

$$(1-h_z)h_x\vec{E} = [h_x\{\vec{E}^{z-}\,\delta_z' + (\frac{\partial \vec{E}}{\partial z})^{z-}\,\delta_z\} - (1-h_z)\{E^{x+}\,\delta_x' + (\frac{\partial \vec{E}}{\partial x})\,\delta_x\}] * K_1 \,. \tag{73}$$

Equations (70)-(73) are formal solutions of (58). The surface quantities $\vec{E}^{z+}$, $(\frac{\partial \vec{E}}{\partial z})^{z+}$, etc. must be determined however, in order to calculate the fields in each region.

A study of the boundary equations (60) - (63) shows that there are eight unknown quantities in (70)-(73), i.e., $h_x \vec{E}^{z+}$, $h_x (\frac{\partial E}{\partial z})^{z+}$, $(1-h_x)\vec{E}^{z+}$, $(1-h_x)(\frac{\partial \vec{E}}{\partial z})^{z+}$, $h_z \vec{E}^{x+}$, $h_z(\frac{\partial \vec{E}}{\partial x})^{x+}$, $(1-h_z)\vec{E}^{x+}$ and $(1-h_z)(\frac{\partial \vec{E}}{\partial x})^{x+}$. We can eliminate two of these, $h_z \vec{E}^{x+}$ and $h_z(\frac{\partial \vec{E}}{\partial x})^{x+}$ quite easily by adding (70) and (71), while noting that (59) requires: $h_z \vec{E}^{x-} = h_z \vec{E}^{x+}$ and $h_z (\frac{\partial \vec{E}}{\partial x})^{x-} = h_z (\frac{\partial \vec{E}}{\partial x})^{x+}$. The result is

$$h_z \vec{E} = [T_{sE}(\vec{J}_s) - h_x \{\vec{E}^{z+}\delta'_z + (\frac{\partial \vec{E}}{\partial z})^{z+}\delta_z\} - (1-h_x)\{\vec{E}^{z+}\delta'_z + (\frac{\partial \vec{E}}{\partial z})^{z+}\delta z\}] * K_0. \quad (74)$$

Referring to (72), (73) and (74), it may readily be seen that the unknown expressions must satisfy the following set of equations:

$$(1-h_z)[[T_{sE}(\vec{J}_s) - h_x \{\vec{E}^{z+}\delta'_z + (\frac{\partial \vec{E}}{\partial z})^{z+}\delta_z\} - (1-h_x)\{\vec{E}^{z+}\delta'_z + (\frac{\partial \vec{E}}{\partial z})^{z+}\delta_z\}] * K_o] = 0 \quad (75)$$

$$\{h_z + h_x(1-h_z)\}[[(1-h_x)\{\vec{E}^{z-}\delta'_z + (\frac{\partial \vec{E}}{\partial z})^{z-}\delta_z\} + (1-h_z)\{\vec{E}^{x-}\delta'_x + (\frac{\partial \vec{E}}{\partial x})^{x-}\delta_x\}] * K_2] = 0 \quad (76)$$

$$\{h_z + (1-h_x)(1-h_z)\}[[h_x\{\vec{E}^{z-}\delta'_z + (\frac{\partial \vec{E}}{\partial z})^{z-}\delta_z\} - (1-h_z)\{\vec{E}^{x+}\delta'_x + (\frac{\partial \vec{E}}{\partial x})^{x+}\delta_x\}] * K_1] = 0 \quad . \quad (77)$$

For convenience, at this point we have not invoked the boundary equations (60) - (63), and so there are six unknown quantities in (75) - (77).

A general solution of (75) - (77) will involve some technique invoking function support arguments, i.e., Wiener-Hopf etc. We shall not attempt this here, but rather, in the next chapter, turn our attention to approximate solutions which should suffice for our purpose.

3. APPROXIMATE SOLUTION FOR THE ELECTRIC FIELD IN THE CASE OF SURFACES WITH SIMPLE DISCONTINUITIES

3.1 Simplifying Assumptions and Resulting Equations

The major problem in seeking solutions to (75) - (77) is the result of the complex convolutions which occur in these equations. Consider (76) and the term

$$(1-h_x)\left\{\vec{E}^{z-}\,\delta_z' + \left(\frac{\partial \vec{E}}{\partial z}\right)^{z-}\,\delta_z\right\} * K_2 \tag{78}$$

Let us carry out a two dimensional (xy) Fourier transform on (78), noting that the convolution with respect to z may be performed immediately. The result is,

$$\underline{(1-h_x)\vec{E}^{z-}}\;\frac{\partial}{\partial z}\left(\frac{e^{-|z|u_2}}{2u_2}\right) + \underline{(1-h_x)\left(\frac{\partial \vec{E}}{\partial z}\right)^{z-}}\;\frac{e^{-|z|u_2}}{2u_2}\,, \tag{79}$$

where the bar beneath a quantity denotes Fourier transform, i.e.

$$\underline{f(k_x,k_y)} = \int_{-\infty}^{\infty}\!\!\int f(x,y)e^{-jk_x x - jk_y y}\,dx,dy,$$

and, $$f(x,y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\!\!\int \frac{\underline{f(k_x,k_y)}}{k_x}\,e^{jk_x x + j\,k_y y}\,dk_x\,dk_y\,.$$

Also

$$\underline{K_2(k_x,k_y)} = \frac{e^{-|z|u_2}}{2u_2},$$

where

$$u_2 = (k_x^2 + k_y^2 - k^2\,n_{02}^2)^{1/2}$$

$$= (\lambda^2 - k^2\,n_{02}^2)^{1/2}\,. \tag{80}$$

The approximation of (79) we shall use is obtained by considering the quantity u_2 in (80) a constant, i.e., independent of λ. This approximation may be given credibility under the assumptions that:

1) $|n_{02}^2| = |\varepsilon_{r2} - j\frac{\sigma_2}{\omega\varepsilon_0}| >> 1$,

2) the important wavenumbers (values of λ) we shall eventually consider are $0 \leq \lambda \leq k$.

Under the conditions (1) and (2), we write

$$u_2 = jk(n_{02}^2 - (\frac{\lambda}{k}))^{1/2} \sim jk\ (n_{02}^2 - (\lambda_0/k)^{1/2}, \text{ where } 0 \leq \lambda_0 \leq k.$$

Now eventually we shall find that λ_0 near k is critical in the sense of asymptotic evaluation of integrals. We therefore write

$$u_2 \sim jk\ (n_{02}^2 - 1)^{1/2} = jkn_{02}^2\ \Delta_2, \tag{81}$$

where, $\Delta_2 = \dfrac{(n_{02}^2 - 1)^{1/2}}{n_{02}^2}$.

The approximation (81) is of the same nature as that used by some other investigators, e.g., Bremmer (1954)

Applying then (81) to (79) gives

$$\frac{(1 - h_x)\vec{E}^{z-}}{} \frac{\partial}{\partial z}\left(\frac{e^{-jkn_{02}^2\ \Delta_2|z|}}{2jk\ n_{02}^2\ \Delta_2}\right) + \frac{(1-h_x)(\frac{\partial\vec{E}}{\partial z})^{z-}}{}\left(\frac{e^{-jkn_{02}^2\ \Delta_2|z|}}{2jkn_{02}^2\ \Delta_2}\right),$$

or taking the inverse transform and the indicated derivative,

$$\{-jk\ n_{02}^2\ \Delta_2\ \text{sgn}\ (z)(1-h_x)\vec{E}^{z-} + (1-h_x)(\frac{\partial\vec{E}}{\partial z})^{z-}\}\ \frac{e^{-jkn_{02}^2\ \Delta_2|z|}}{2jkn_{02}^2\ \Delta_2}, \tag{82}$$

where

$$\text{sgn}(z) = \begin{cases} 1, & z > 0 \\ -1, & z < 0 \end{cases} .$$

We shall regard (82) as a first estimate of (78). Further, since the source is assumed to be located above the z=0 plane, any excitation for the term

$$(1-h_z)\{\vec{E}^{x-}\delta_x' + (\frac{\partial \vec{E}}{\partial z})^{x-}\ \delta_x\} * K_2 \tag{83}$$

in (76) must originate with the term (78) through equation (75). Referring to (82) it may be seen that if

$$n_{02} = (\varepsilon_{r2} - j\frac{\sigma_2}{\omega\varepsilon_0})^{1/2} = \beta - j\alpha$$

has a large negative imaginary part, i.e., if α is large or the media is highly conductive, then $(1-h_z)\vec{E}^{x-}$ and $(1-h_x)(\frac{\partial \vec{E}}{\partial z})^{x-}$ in (83) are small. We therefore take zero as a first estimate of (83) in (76).

Under these assumptions and using (82), we take (76) as

$$\{h_z + h_x(1-h_z)\}[-jkn_{02}^2\ \Delta_2 \text{sgn}(z)(1-h_x)\vec{E}^{z-} + (1-h_x)(\frac{\partial \vec{E}}{\partial z})^{z-}] \cdot \frac{e^{-jkn_{02}^2\ \Delta_2|z|}}{2jkn_{02}^2\ \Delta_2} = 0. \tag{84}$$

(84) is satisfed if,

$$(1-h_x)(\frac{\partial \vec{E}}{\partial z})^{z-} = jkn_{02}^2\Delta_2\ (1-h_x)\vec{E}^{z-} . \tag{85}$$

We may now use (72) to obtain a second estimate of $(1-h_z)(\frac{\partial \vec{E}}{\partial x})^{x-}$ and $(1-h_z)\vec{E}^{x-}$ for (83). Using (82) and (85), our first estimate of (83), (72) may be written as,

$$(1-h_z)(1-h_x)\vec{E} = (1-h_x)\vec{E}^{z-}(x,y)\ e^{-jkn_{02}^2\Delta_2|z|}$$

$$(1-h_z)(1-h_x)\frac{\partial \vec{E}}{\partial x} = (1-h_x)\ \frac{\partial \vec{E}^{z-}}{\partial x}\ (x,y)\ e^{-jkn_{02}^2\Delta_2|z|},$$

so

$$(1-h_z)\vec{E}^{x-} = (\vec{E}^{z-})^{x-}\ e^{-jkn_{02}^2\Delta_2|z|} \tag{86}$$

$$(1-h_z)(\frac{\partial \vec{E}}{\partial x})^{x-} = (\frac{\partial \vec{E}^{z-}}{\partial x})^{x-}\ e^{-jkn_{02}^2\Delta_2|z|}\ . \tag{87}$$

Equations (86) and (87) indicate that the error in our first estimate of $(1-h_z)\vec{E}^{x-}$ and $(1-h_z)(\frac{\partial \vec{E}}{\partial x})^{x-}$ is of negative exponential order with respect to the fields immediately below the surface. We shall eventually be concerned mainly with the vertical or z-component of the fields and note that (62) and (63) require

$$(E_z^{z-})^{x-} = 0$$

$$(\frac{\partial E_z^{z-}}{\partial x})^{x-} = \frac{1}{n_{02}^2}\ (\frac{\partial E_z^{z+}}{\partial x})^{x-}\ .$$

We therefore assume that (85) is sufficiently accurate, under the assumptions stated above, although it is strictly true only for a perfect conductor.

A similar argument applied to (77) and (73) gives

$$h_x\left(\frac{\partial \vec{E}}{\partial z}\right)^{z-} = jkn_{01}^2 \Delta_1 h_x \vec{E}^{z-} \tag{88}$$

where

$$\Delta_1 = \frac{(n_{02}^2-1)^{1/2}}{n_{01}^2}, \quad n_{01}^2 = \varepsilon_{r1} - j\frac{\sigma_1}{\omega\varepsilon_0}.$$

It is our intention now to use (85), (88) and (75) to derive an equation for the expression for $\vec{E}^{z+}$, the electric surface field. We first write (85) and (88) in terms of expressions with positive superscripts as required by (75). Refer to the boundary equations (62), we have

$$(1-h_x)\vec{E}^{z-} = (1-h_x)\left[\vec{E}^{z+} - \frac{n_{02}^2 - 1}{n_{02}^2} E_z^{z+} \vec{a}_z\right]$$

$$(1-h_x)\left(\frac{\partial \vec{E}}{\partial z}\right)^{z-} = (1-h_x)\left[\left(\frac{\partial \vec{E}}{\partial z}\right)^{z+} - \frac{n_{02}^2 - 1}{n_{02}^2} \nabla_{xy} (E_z^{z+})\right] .$$

Hence (85) may be written as,

$$(1-h_x)\left[\left(\frac{\partial \vec{E}}{\partial z}\right)^{z+} - \frac{n_{02}^2 - 1}{n_{02}^2} \nabla_{xy} (E_z^{z+})\right]$$

$$= jkn_{02}^2 \Delta_2 (1-h_x)\left[\vec{E}^{z+} - \frac{n_{02}^2 - 1}{n_{02}^2} E_z^{z+} \vec{a}_z\right],$$

or

$$(1-h_x)\left(\frac{\partial \vec{E}}{\partial z}\right)^{z+} = \frac{n_{02}^2 - 1}{n_{02}^2} \nabla_{xy} \left[(1-h_x)E_z{}^{z+}\right]$$

$$+ jkn_{02}^2\Delta_2 \, (1 - h_x) \left[\vec{E}^{z+} - \frac{n_{02}^2 - 1}{n_{02}^2} E_z{}^{z+} \hat{a}_z\right], \qquad (89)$$

where, we note that

$$(1-h_x) \nabla_{xy} (E_z{}^{z+}) = \nabla_{xy}\left[(1-h_x)E_z{}^{z+}\right] ,$$

since by (63b),

$$(E_z{}^{z+})^{x-} = 0 \quad .$$

(89) is conveniently written in matrix form as

$$(1-h_x)\overline{D_z E^{z+}} = A_2 \, (1-h_x)\overline{E^{z+}} \qquad (90)$$

where

$$\overline{D_z E^{z+}} = \begin{bmatrix} \left(\frac{\partial E_x}{\partial z}\right)^{z+} \\ \left(\frac{\partial E_y}{\partial z}\right)^{z+} \\ \left(\frac{\partial E_z}{\partial z}\right)^{z+} \end{bmatrix}, \qquad \overline{E^{z+}} = \begin{bmatrix} E_x{}^{z+} \\ E_y{}^{z+} \\ E_z{}^{z+} \end{bmatrix}$$

$$A_2 = \begin{bmatrix} jkn_{02}^2\Delta_2 & 0 & \frac{n_{02}^2 - 1}{n_{02}^2} \frac{\partial}{\partial x} \\ 0 & jkn_{02}^2\Delta_2 & \frac{n_{02}^2 - 1}{n_{02}^2} \frac{\partial}{\partial y} \\ 0 & 0 & jk\Delta_2 \end{bmatrix}$$

Also, (88) may be written as,

$$h_x \overline{D_z E^{z+}} = A_1 \, h_x \overline{E^{z+}}, \tag{91}$$

where $\overline{D_z E^{z+}}$ and $\overline{E^{z+}}$ are the vectors defined in (90) and,

$$A_1 = \begin{bmatrix} jkn_{01}^2 \Delta_1 & 0 & \dfrac{n_{01}^2 - 1}{n_{02}^2} \dfrac{\partial}{\partial y} \\ 0 & jkn_{01}^2 \Delta_1 & \dfrac{n_{01}^2 - 1}{n_{01}^2} \dfrac{\partial}{\partial y} \\ 0 & 0 & jk\Delta_1 \end{bmatrix}.$$

Refer now to (75). It is convenient, because of the convolution form of that equation, to take two-dimensional (x,y) Fourier transforms. Accordingly, that equation implies

$$(1-h_z)\Big[\underline{\overset{+}{E}_s} - \Big\{uh_x \underline{\overset{+}{E}^{z+}} + h_x \Big(\frac{\partial \overset{+}{E}}{\partial z}\Big)^{z+}\Big\} - \Big\{u\,(1-h_x)\underline{\overset{+}{E}^{z+}} + (1-h_x)\Big(\frac{\partial \overset{+}{E}}{\partial z}\Big)^{z+}\Big\}\Big] \frac{e^{-|z|u}}{2u} = 0 \tag{92}$$

where the notation for Fourier transform is consistant with (79).

Also,

$$\underline{K_0} = \frac{e^{-|z|u}}{2u}$$

$$u = (k_x^2 + k_y^2 - k^2)^{1/2} = (\lambda^2 - k^2)^{1/2}$$

$$\underline{\overset{+}{E}_s} = \underline{T_{sE}(\overset{+}{J}_s) * K_o} = \underline{T_{sE}(\overset{+}{J}_s)} \overset{z}{*} \frac{e^{-|z|u}}{2u},$$

where $\overset{z}{*}$ denotes convolution with respect to z.

Choosing a plane, say $z^- < 0$, and using (90) and (91), we have

$$2u \underline{\bar{E}_s}^{z^-} e^{|z^-|u} = (uI + \underline{A_1})\, \underline{h_x \bar{E}^{z+}} + (uI + \underline{A_2})\underline{(1-h_x)\bar{E}^{z+}} \qquad (93)$$

where

$$\underline{\bar{E}}_s^{z^-} = \begin{bmatrix} \underline{E}_{sx}^{z^-} \\ \underline{E}_{sy}^{z^-} \\ \underline{E}_{sz}^{z^-} \end{bmatrix}$$

is the transform of the source vector evaluated in the plane $z=z^-$.

Also I is the 3x3 identity matrix and,

$$\underline{A_1} = \begin{bmatrix} jkn_{01}^2\Delta_1 & 0 & jk_x \dfrac{n_{01}^2 - 1}{n_{01}^2} \\ 0 & 0 & jk_y \dfrac{n_{01}^2 - 1}{n_{01}^2} \\ 0 & 0 & jk\Delta_1 \end{bmatrix}$$

$$\underline{A_2} = \begin{bmatrix} jkn_{02}^2\Delta_2 & 0 & jk_x \dfrac{n_{02}^2 - 1}{n_{02}^2} \\ 0 & jkn_{02}^2\Delta_2 & jk_y \dfrac{n_{02}^2 - 1}{n_{02}^2} \\ 0 & 0 & jk\Delta_2 \end{bmatrix}$$

$$\underline{h_x \bar{E}^{z+}} = \begin{bmatrix} \underline{h_x E_x{}^{z+}} \\ h_x E_y{}^{z+} \\ \underline{h_x E_z{}^{z+}} \\ x \end{bmatrix}, \qquad \underline{(1-h_x)\bar{E}^{z+}} = \begin{bmatrix} \underline{(1-h_x)E_x{}^{z+}} \\ \underline{(1-h_x)E_y{}^{z+}} \\ (1-h_x)E_z{}^{z+} \end{bmatrix}.$$

Noting that

$$\underline{(1-h_x)\bar{E}^{z+}} = \underline{\bar{E}^{z+}} - \underline{h_x \bar{E}^{z+}},$$

(93) may be rewritten as,

$$(uI + \underline{A_2})\underline{\bar{E}^{z+}} - (\underline{A_2} - \underline{A_1})\underline{h_x \bar{E}^{z+}} = 2ue^{|z-|u}\, \underline{\bar{E}_x{}^{z-}}$$

or

$$\underline{\bar{E}^{z+}} - (uI + \underline{A_2})^{-1}(\underline{A_2} - \underline{A_1})\, \underline{h_x \bar{E}^{z+}} = \underline{\bar{E}_{0s}}\ , \tag{94}$$

where

$$\underline{\bar{E}_{0s}} = 2ue^{|z-|u}\ (uI + \underline{A_2})^{-1}\, \underline{\bar{E}_s{}^{z-}}\ .$$

For convenience, let us compute $(uI + \underline{A_2})^{-1}$. Now

$$uI + \underline{A_2} = \begin{bmatrix} u + jkn_{02}^2\Delta_2 & 0 & jk_x \dfrac{n_{02}^2 - 1}{n_{02}^2} \\ 0 & u + jkn_{02}^2\Delta_2 & jk_y \dfrac{n_{02}^2 - 1}{n_{02}^2} \\ 0 & 0 & u+jk\Delta_2, \end{bmatrix},$$

hence

$$(u\underline{I} + \underline{A}_2)^{-1} = \begin{bmatrix} \dfrac{1}{u+jkn_{02}^2\Delta_2} & 0 & \dfrac{-jk_x\dfrac{n_{02}^2-1}{n_{02}^2}}{(u+jkn_{02}^2\Delta_2)(u+jk\Delta_2)} \\ 0 & \dfrac{1}{u+jkn_{02}^2\Delta_2} & \dfrac{-jk_y\dfrac{n_{02}^2-1}{n_{02}^2}}{(u+jkn_{02}^2\Delta_2)(u+jk\Delta_2)} \\ 0 & 0 & \dfrac{1}{u+jk\Delta_2} \end{bmatrix}$$

Also,

$$(u\underline{I} + \underline{A}_2)^{-1}(\underline{A}_2 - \underline{A}_1)) = \underline{G} \text{ (say)} =$$

$$\begin{bmatrix} \dfrac{jk(n_{02}^2\Delta_2 - n_{01}^2\Delta_1)}{u+jkn_{02}^2\Delta_2} & 0 & \dfrac{jk_x(\frac{1}{n_{01}^2} - \frac{\Delta_1}{n_{02}^2})}{u+jkn_{02}^2\Delta_2} - \dfrac{jk_y(\Delta_2-\Delta_1)}{n_{02}^2\Delta_2(u+jk\Delta_2)} \\ 0 & \dfrac{jk(n_{02}^2\Delta_2 - n_{01}^2\Delta_1)}{u+jkn_{02}^2\Delta_2} & \dfrac{jk_x(\frac{1}{n_{01}^2} - \frac{\Delta_1}{n_{02}^2\Delta_2})}{u+jkn_{02}^2\Delta_2} - \dfrac{jk_y(\Delta_2-\Delta_1)}{n_{02}^2\Delta_2(u+jk\Delta_2)} \\ 0 & 0 & \dfrac{jk(\Delta_2-\Delta_1)}{u+jk\Delta_2} \end{bmatrix} \tag{96}$$

Therefore (94) may be written as,

$$\bar{\underline{E}}^{z+} - \underline{G}\,\underline{h_x}\bar{\underline{E}}^{z+} = \bar{\underline{E}}_{0s} \quad , \tag{97}$$

where $\bar{\underline{E}}_{0s}$ is defined in (94) and $\underline{G}$ is the matrix (96).

Equation (97) is the equation which must be solved to determine an expression for the surface field $\bar{\underline{E}}^{z+}$.

3.2 A Series Solution for the Electric Surface Field

Refer to (97) and for simplicity, initially consider the vertical component of the electric field, i.e., $\underline{E}_z^{z+}$. Equation (96) shows that this component is not coupled to the other components and so the relevant equation may be written as

$$\underline{E}_z^{z+} - \underline{g}_z \, \underline{h_x E_z}^{z+} = \underline{E}_{0sz}, \tag{98}$$

where

$$\underline{g}_z = \frac{jk(\Delta_2 - \Delta_1)}{u+jk\Delta_2}$$

$$\underline{E}_{0sz} = \frac{2ue^{|z-|u} \, \underline{E_{sz}}^{z-}}{u+jk\Delta_2}.$$

Let us write

$$\underline{E_z}^{z+} = \underline{h_x E_z}^{z+} + \underline{(1-h_x)E_z}^{z+}$$

$$\underline{g_z} = \underline{h_x g_z} + \underline{(1-h_x)g_z}$$

$$\underline{E_{0sz}} = \underline{h_x E_{0sz}} + \underline{(1-h_x)E_{0sx}} \quad .$$

Hence, (98) may be written as,

$$\underline{h_x E_z}^{z+} + \underline{(1-h_x)E_z}^{z+} - \left[\underline{h_x g_z} + \underline{(1-h_x)g_z}\right] \underline{h_x E_z}^{z+}$$

$$= \underline{h_x E_{0sz}} + \underline{(1-h_x)E_{0sz}} \, . \tag{99}$$

Equation (99) implies,

$$\underline{h_x E_z}^{z+} - \underline{h_x g_z} \, \underline{h_x E_z}^{z+} - F_x \left\{h_x F_x^{-1} \left[\underline{(1-h_x)g_z} \, \underline{h_x E_z}^{z+}\right]\right\} = \underline{h_x E_{0sz}},$$

and,

$$\underline{(1-h_x)E_z}^{z+} - F_x\{(1-h_x)F_x^{-1}\,[\underline{(1-h_x)g_z}\ \underline{h_xE_z}^{z+}]\} = \underline{(1-h_x)E_{osz}}$$

or,

$$\underline{h_xE_z}^{z+} - \frac{1}{1-\underline{h_xg_z}} F_x\{h_xF_x^{-1}\,[\underline{(1-h_x)g_z}\ \underline{h_xE_z}^{z+}]\}$$

$$= \frac{1}{1-\underline{h_xg_z}}\ \underline{h_x\,E_{osz}} \qquad (100)$$

$$\underline{(1-h_x)E_z}^{z+} = \underline{(1-h_x)E_{osz}} + F_x\{(1-h_x)F_x^{-1}\,[\underline{(1-h_x)g_x}\ \underline{h_xE_z}^{z+}]\}, \qquad (101)$$

where $F_x(F_x^{-1})$ denotes one dimension Fourier (inverse Fourier) transform, i.e.

$$F_x^{-1}\,[\underline{f(k_x,k_y)}] = \frac{1}{2\pi}\int_{kx} \underline{f(k_y,k_y)}\ e^{jk_xx}dk_x$$

$$= \underline{f(x,k_y)}$$

$$F_x[f(x,k_y)] = \int_x \underline{f(x,k_y)}e^{-jk_xx}dx$$

$$= \underline{f(k_x,\ k_y)}$$

Equation (100) is the basic equation, whose solution $\underline{h_xE_x}^{z+}$ represents the vertical component of the surface electric field propagated beyond the discontinuity located at x=0. Recall we are assuming the source to be located to the left of the discontinuity i.e. in the region x<0. The left side of (101) represents the vertical electric surface field to the left of the discontinuity, i.e., in the region in which the source is assumed to be located. It should be noted that the right side of (101) may be evaluated if $\underline{h_xE_z}^{z+}$ is known.

We now consider (100) and note that this equation may be written in the form

$$(I-T)\underline{h_x E_z}^{z+} = \underline{h_x E}, \tag{102}$$

where I is the identity operator and T is a linear operator defined from the second term on the left side of (100), i.e.,

$$T\,(\underline{h_x E_z}^{z+}) = \frac{1}{\underline{1-h_x g_z}}\, F_x\{h_x F_x^{-1}[\underline{(1-h_x)g_z}\ (\underline{h_x E_z}^{z+})]\}$$

and

$$\underline{h_x E} = \frac{1}{\underline{1-h_x g_z}}\ \underline{h_x E_{osz}}.$$

A formal solution of (102) may be written as

$$\underline{h_x E_z}^{z+} = \underline{h_x E} + T(\underline{h_x E}) + T^2(\underline{h_x E}) + \ldots.$$

$$= \sum_{n=0}^{\infty} T^n(\underline{h_x E}) \quad . \tag{103}$$

We first consider the term

$$\frac{1}{\underline{1-h_x g_z}} = 1 + \frac{\underline{h_x g_z}}{\underline{1-h_x g_z}}\ , \tag{104}$$

occurring in (100) and (102). We shall assume at this point

$$\frac{\underline{h_x g_z}}{\underline{1-h_x g_z}} \simeq \underline{h_x F_x^{-1}\{\frac{g_z}{1-g_z}\}} = \underline{h_x g_z'} \tag{105}$$

where from (98)

$$\underline{g_z}' = \frac{\frac{jk(\Delta_2-\Delta_1)}{u + j\,k\,\Delta_2}}{1 - \frac{jk(\Delta_2-\Delta_1)}{u + j\,k\,\Delta_2}}$$

$$= \frac{j\,k\,(\Delta_2-\Delta_1)}{u + j\,k\,\Delta_1}. \qquad (106)$$

Thus (104) may be written as

$$\frac{1}{1 - \underline{h_x g_z}} \simeq 1 + \underline{h_x g_z'} \quad . \qquad (107)$$

3.3 The First Order Forward Propagated (x>0) and Backscattered (x<0 Vertical Electric Field

Referring to (103), we shall refer to the approximation

$$\left[\underline{h_x E_z}^{z+}\right]_1 = \underline{h_x E} \qquad (108)$$

the first order solution for the forward propagated vertical field. Similarly, referring to (101), we shall call

$$\left[\underline{(1-h_x)E_z}^{z+}\right]_1 = \underline{(1-h_x)E_{osz}} + F_x\{1-h_x)F_x^{-1}[\underline{(1-h_x)g_z}\ \underline{h_x E}]\} \qquad (109)$$

the first order solution for the vertical field in the region x<0. The second term in (109) we shall refer to as the first order backscattered vertical electric field, since it represents a modification in the vertical field for x<0 arising because of the discontinuity at x=0.

Using (102), (107), (98)

$$[\underline{h_x E_z}^{z+}]_1 = \underline{h_x E} = \frac{1}{1-\underline{h_x g_z}}\,\underline{h_x E_{osz}}$$

$$\simeq \underline{h_x E_{osz}} + \underline{h_x g'_z}\,\underline{h_x E_{osz}}$$

$$= \{F_x h_x F_x^{-1} \left(\frac{2u\, e^{|z^-|u}\, \underline{E}^{z^-}_{sz}}{u + j\,k\,\Delta_2}\right)\}$$

$$+ \{[F_x h_x F_x^{-1}\left(\frac{jk\,(\Delta_2-\Delta_1)}{u + j\,k\,\Delta}\right)][F_x h_x F_x^{-1}\left(\frac{2u\, e^{|z^-|u}\, \underline{E}^{z^-}_{sz}}{u + j\,k\,\Delta_2}\right)]\}\ , \tag{110}$$

where $\underline{E}^{z^-}_{sz}$ is defined in (93), i.e.,

$$\underline{E}^{z^-}_{sz} = \left(\underline{T_{SE}}(\vec{J}_s) \overset{z}{*} \frac{e^{-|z|u}}{2u}\right)_{z\,=\,z^-}\ .$$

So in (110)

$$\frac{2ue^{|z^-|u}\, \underline{E}^{z^-}_{sz}}{u + j\,k\,\Delta_2} = \frac{e^{|z^-|u}\, \left(T_{SE}(\vec{J}_s)_z \overset{z}{*} e^{-|z|u}\right)_{z=z^-}}{u + j\,k\,\Delta_2}\ , \tag{111}$$

$T_{SE}(\vec{J}) = \frac{1}{j\omega\varepsilon_o}\left[\nabla(\nabla\cdot\vec{J}_s) + k^2\,\vec{J}_s\right]$ from (30).

We shall now show that (110) agrees, after a stationary phase integration, with the result derived by Bremmer (1954) when the source current $\vec{J}_s$ is that of a vertical dipole located at the coordinates $(-x_0,\ 0,\ a)$, $x_0,\ a > 0$. Accordingly, adopting standard notation, let

$$\vec{J}_s = I\ \Delta\ \ell\ \delta(x+x_0)\ \delta(y)\ \delta(z-a)\ \vec{a}_z . \tag{112}$$

Then

$$T_{SE}(\vec{J}_s)_z = \frac{1}{j\omega\varepsilon_0}\left[\nabla(\nabla\cdot\vec{J}_s) + k^2\vec{J}_s\right]_z$$

$$= \frac{I\ \Delta\ \ell}{j\omega\varepsilon_0}\left[\delta''(z-a) + k^2\delta(z-a)\right]\ \delta(x+x_0)\ \delta(y)$$

and

$$\underline{T_{SE}(\vec{J}_s)_z} = \frac{I\ \Delta\ \ell}{j\omega\varepsilon_0}\left[\delta''(z-a) + k^2\delta(z-a)\right]e^{jk_x x_o} . \tag{113}$$

Therefore

$$\left(\underline{T_{SE}(\vec{J}_s)_z}\right) \overset{z}{*} e^{-|z|u}\Big)_{z=z^-} = \left[\frac{I\ \Delta\ \ell}{j\omega\varepsilon_0}\ e^{jk_x x_o}\ \left(\frac{\partial^2}{\partial z^2} + k^2\right)\ e^{-|z-a|u}\right]_{z=z^-}$$

$$= \frac{I\ \Delta\ \ell}{j\omega\varepsilon_0}\ e^{jk_x x_o}\ (u^2+k^2)\ e^{z^- u}\ e^{-au} \tag{114}$$

since $z^- < 0$. Therefore from (111),

$$\frac{2\ u\ e^{|z^-|u}\ \underline{E}^{z^-}_{sz}}{u + j\ k\ \Delta_2} = \frac{I\ \Delta\ \ell}{j\omega\varepsilon_0}\ e^{jk_x x_0}\ \frac{(u^2+k^2)\ e^{-au}}{u + j\ k\ \Delta_2}$$

$$= \frac{I\ \Delta\ \ell}{j\omega\varepsilon_0}\ \frac{\lambda^2 e^{-au}}{u + j\ k\ \Delta_2}\ e^{jk_x x_0} , \tag{115}$$

$\lambda^2 = k_x^2 + k_y^2$. We shall now write (110) in the form of a convolution in the real spatial coordinates x and y. We must therefore consider the inverse Fourier transforms of the functions involved.

Let us consider,

$$F_{xy}^{-1}\left[\frac{1}{u + j\,k\,\Delta_2}\right] = \frac{1}{(2\pi)^2}\int_{k_x}\int_{k_y}\frac{1}{u + j\,k\,\Delta_2}\,e^{jk_x x + jk_y y}\,dk_x dk_y$$

$$= \frac{1}{2\pi}\int_0^\infty \frac{\lambda}{u + j\,k\,\Delta_2}\,J_0(\lambda\rho)\,d\lambda\,, \qquad (116)$$

where $\rho = (x^2+y^2)^{1/2}$. The integral appearing in (116) has been a topic of discussion for many years (see Wait, 1970). The form which we shall use is essentially that proposed by Norton (1936, 1937), and is an asymptotic form valid when $k\rho \gg 1$. We therefore write

$$F_{xy}^{-1}\left[\frac{1}{u + j\,k\,\Delta_2}\right] \sim \frac{1}{2\pi}\,F_2(\rho)\,\frac{e^{-jk\rho}}{\rho}\,, \qquad (117)$$

where,

$$F_2(\rho) = 1 - j\sqrt{\pi\,p_2}\;e^{-p_2}\;\mathrm{erfc}\,(j\sqrt{p_2})$$

$$p_2 = -j\,k\,\frac{\rho}{2}\,\Delta_2^2\,.$$

Similarly, we take

$$F_{xy}^{-1}\left[\frac{1}{u + j\,k\,\Delta_1}\right] \simeq \frac{1}{2\pi}\,F_1(\rho)\,\frac{e^{-jk\rho}}{\rho} \qquad (118)$$

where,

$$F_1(\rho) = 1 - j\sqrt{\pi p_1}\, e^{-p_1}\, \mathrm{erfc}\,(j\sqrt{p_1})$$

$$p_1 = -j\,\frac{k\rho}{2}\,\Delta_1^{\,2} \quad .$$

Using (117), we may write the inverse transform of (115) as

$$F_{xy}^{-1}\left[\frac{I\,\Delta\,\ell}{j\omega\varepsilon_0}\,\frac{\lambda^2 e^{-au}}{u+j\,k\,\Delta_2}\,e^{jk_x x_0}\right]_{a\to 0}$$

$$= \frac{I\,\Delta\,\ell}{j\omega\varepsilon_0}\,\frac{1}{2\pi}\left[-\nabla_{xy}^2\left\{F_2(\rho_0)\,\frac{e^{-jk\rho_0}}{\rho_0}\right\}\right] , \qquad (119)$$

where

$$\rho_0 = ((x+x_0)^2 + y^2)^{1/2}$$

$$\nabla_{xy}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad .$$

Since we have $k\rho_0 >> 1$ and $|\Delta_2| << 1$ we may take as a far field estimate of (119)

$$\nabla_{xy}^{\,2}\left\{F_2(\rho_0)\,\frac{e^{-jk\rho_0}}{\rho_0}\right\} \simeq -\,k^2\,\frac{e^{-jk\rho_0}}{\rho_0} ,$$

hence,

$$F_{xy}^{-1}\left[\frac{I\,\Delta\,\ell}{j\omega\varepsilon_0}\,\frac{\lambda^2}{u+jk\Delta_2}\,e^{jk_x x_0}\right] \simeq \frac{I\,\Delta\,\ell\,k^2}{j\omega\varepsilon_0(2\pi)}\,F_2(\rho_0)\,\frac{e^{-jk\rho_0}}{\rho_0} \quad . \qquad (120)$$

Using then (118) and (120), (110) may be written in the form,

$$[h_x E_z^{z+}] = \frac{I \Delta \ell k^2}{j\omega\varepsilon_0(2\pi)} h_x \left[F_2(\rho_0) \frac{e^{-jk\rho_0}}{\rho_0}\right.$$

$$\left. + \frac{jk(\Delta_2-\Delta_1)}{2\pi} \int_{y'} \int_{0}^{x} F_1(\rho_a)\, F_2(\rho_b) \frac{e^{-jk(\rho_a+\rho_b)}}{\rho_a \rho_b} \right.$$

$$\left. dx'dy' \right] \qquad (121)$$

where

$$\rho_a = ((x-x')^2 + (y-y')^{1/2}$$

$$\rho_b = ((x'+x_0)^2 + y'^2)^{1/2}.$$

The double integral in (121) may be estimated by a single integral by means of a stationary phase integration (Hufford, 1952, Bremmer, 1954, King, 1969).

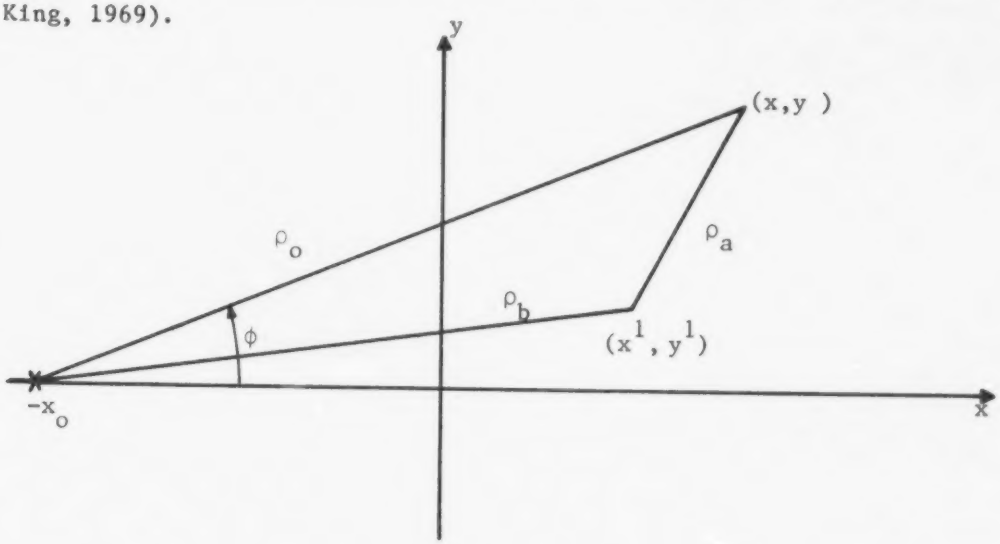


Figure 6. Variables for Stationary Phase Integration of (121).

To this end we introduce the elliptic coordinates

$$\rho_b = \frac{\rho_0}{2}\left[\cosh v + \cos\theta\right]$$

$$\rho_a = \frac{\rho_0}{2}\left[\cosh v - \cos\theta\right] .$$

The integral in (121) therefore becomes (see, for example, King, 1969, for details),

$$\int_v \int_{\theta(v)} F_1\left[\frac{\rho_0}{2}(\cosh v - \cos\theta)\right] F_2\left[\frac{\rho_0}{2}(\cosh v + \cos\theta)\right] e^{-jk\rho_0 \cosh v} \, dv \, d\theta \quad . \quad (122)$$

The stationary phase point is at $v = 0$, hence assuming F_1 and F_2 are slowly varying and $k\rho_0 \gg 1$, we have (122) as

$$\int_0^{\theta(0)} F_1\left[\frac{\rho_0}{2}(1-\cos\theta)\right] F_2\left[\frac{\rho_0}{2}(1+\cos\theta)\right] \int_{-\infty}^{\infty} e^{-jk\rho_0\left(1+\frac{v^2}{2}\right)} \, dv \, d\theta$$

$$= \left(\frac{2\pi}{jk}\right)^{1/2} \frac{e^{-jk\rho_0}}{\sqrt{\rho_0}} \int_0^{\theta(0)} F_1\left[\frac{\rho_0}{2}(1-\cos\theta)\right] F_2\left[(1+\cos\theta)\right] d\theta \quad . \quad (123)$$

Setting

$$\rho' = \frac{\rho_0}{2}(1+\cos\theta) \ ,$$

then (123) may be written as,

$$\left(\frac{2\pi}{jk}\right)^{1/2} \frac{e^{-jk\rho_0}}{\sqrt{\rho_0}} \int_{x_0 \sec\phi}^{\rho_0} \frac{F_1(\rho_0-\rho')F_2(\rho')}{(\rho'(\rho_0-\rho'))^{1/2}} \, d\rho' \quad . \quad (124)$$

Hence (121) becomes

$$[h_x E_z^{z+}]_1 = h_x \{\frac{I \Delta \ell k^2}{2\pi j\omega\varepsilon_0} \frac{e^{-jk\rho_0}}{\rho_0} [F_2(\rho_0)$$

$$+ (\frac{jk}{2\pi})^{1/2} (\Delta_2-\Delta_1) \sqrt{\rho_0} \int_{x_0 \sec\phi}^{\rho_0} \frac{F_1(\rho_0-\rho')F_2(\rho')}{(\rho'(\rho_0-\rho'))^{1/2}} d\rho']\} \quad . \qquad (125)$$

It may easily be verified that (125) agrees with the expression for the field propagated beyond the boundary as developed by Bremmer (1954) (see also Wait, 1962, 1974). It should be emphasized that although we have used the vertical dipole and stationary phase integration in arriving at (125), (110) is valid for any finite antenna system expressible as in (111).

Let us now return to (109), the first order expression for the vertical electric field in the region $x<0$. Let us write this out using the results of (110)

$$[\underline{(1-h_x)E_z^{z+}}]_1 = F_x[(1-h_x)F_x^{-1}(\frac{2u\, e^{|z^-|u_E}}{u + j\,k\,\Delta_2}\underline{_{sz}}\, z^-)]$$

$$+ F_x[(1-h_x)F_x^{-1}\{F_x[(1-h_x)F_x^{-1}(\frac{jk(\Delta_2-\Delta_1)}{u + j\,k\,\Delta_2})]$$

$$\cdot \underline{(h_x E_z^{z+})}_1\}] \qquad (126)$$

where $\underline{(h_x E_z^{z+})}_1$ is given by (110). Equation (126) may also be written in terms of inverse transforms and convolutions as,

$$[(1-h_x)E_z^{z+}]_1 = (1-h_x)\, F_{xy}^{-1}(\frac{2u\, e^{|z^-|u_E}}{u + j\,k\,\Delta_2}\underline{_{sz}}\, z^-)$$

$$+ (1-h_x)[(1-h_x)\, F_{xy}^{-1}(\frac{jk(\Delta_2-\Delta_1)}{u + j\,k\,\Delta_2}) \underset{xy}{*} (h_x E_z^{z+})_1] \quad . \qquad (127)$$

Using (120) and (121), the expression for the vertical dipole (127) becomes,

$$\left[(1-h_x)E_z{}^{z+}\right]_1 = (1-h_x)\frac{I\,\Delta\,\ell\,k^2}{j\omega\varepsilon_0(2\pi)}\left[F_2(\rho_0)\frac{e^{-jk\rho_0}}{\rho_0}\right.$$

$$+\frac{jk(\Delta_2-\Delta_1)}{2\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty}F_2(\rho_c)\frac{e^{-jk\rho_c}}{\rho_c}\left\{F_2(\rho_0')\frac{e^{-jk\rho_0'}}{\rho_0'}\right.$$

$$+\frac{jk(\Delta_2-\Delta_1)}{2\pi}\int_{-\infty}^{\infty}\int_{0}^{x'}F_1(\rho_a)\,F_2(\rho_b)\,\frac{e^{-jk(\rho_a+\rho_b)}}{\rho_a\rho_b}$$

$$\left.\left.dx''\,dy''\right\}dx'\,dy'\right] \qquad (128)$$

where

$$\rho_0 = ((x+x_0)^2+y^2)^{1/2}$$

$$\rho_a = ((x'+x_0)^2 + (y'-y'')^2)^{1/2}$$

$$\rho_b = ((x''+x_0)^2 + y''^2)^{1/2}$$

$$\rho_c = ((x-x')^2 + (y-y')^2)^{1/2}$$

$$\rho_0' = ((x'+x_0)^2 + y'^2)^{1/2}.$$

If we employ the stationary phase integration (125), then (128) may be seen to be,

$$\left[1-h_x)E_z{}^{z+}\right]_1 = (1-h_x)\frac{I\,\Delta\,\ell\,k^2}{j\omega\varepsilon_0(2\pi)}\left[F_2(\rho_0)\frac{e^{-jk\rho_0}}{\rho_0}\right.$$

$$+\frac{jk(\Delta_2-\Delta_1)}{2\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty}F_2(\rho_c)\,\frac{e^{-jk(\rho_c+\rho_0')}}{\rho_c\rho_0'}\left\{F_2(\rho_0')\right.$$

$$\left.+(\Delta_2-\Delta_1)\left(\frac{jk\rho_0'}{2\pi}\right)^{1/2}\int_{x_0\sec\phi}^{\rho_0'}\frac{F_1(\rho_0'-\rho')F_2(\rho')}{(\rho'(\rho_0'-\rho'))^{1/2}}\,d\rho'\right\}$$

$$\left.dx'\,dy'\right]. \qquad (126)$$

The physical interpretation of (126)(and (128)) is clear. The first term is the field which would exist for $x<0$ if the medium was homogenous. The second term is a modification to the first, resulting from the fact that the outward going wave changes after it enters the region $x>0$, as indicated by (121) and (125). This modification we will refer to as the backscattered field.

3.4 Higher Order Terms for the Forward Propagated Vertical Electric Field

Let us consider the second term in the series (103).

$$T(\underline{h_x E}) = \frac{1}{1-\underline{h_x g_z}} \left[\underline{h_x\{(1-h_x)g_z * h_x E\}}\right]$$

$$\cdot$$

$$= (1+\underline{h_x g_z'}) \left[\underline{h_x\{(1-h_x)g_z * h_x E\}}\right] , \tag{127}$$

where * denotes 2-dimensional (x,y) convolution,

$$g_z = F_{xy}^{-1}\left[\frac{jk(\Delta_2-\Delta_1)}{u + j\,k\,\Delta_2}\right]$$

$$g_z' = F_{xy}^{-1}\left[\frac{jk(\Delta_2-\Delta_1)}{u + j\,k\,\Delta_1}\right]$$

$$h_x E = h_x F_{xy}^{-1}\left[\frac{1}{1-\underline{h_x g_z}}\,\underline{h_x E_{osz}}\right]$$

$$\cdot$$

$$= h_x\, F_{xy}^{-1}\left[(1+\underline{h_x g_z'})\,\underline{h_x E_{osz}}\right] \tag{128}$$

and $\underline{E_{osz}}$ is given in (111). We now write out (127) in the form of integrals. Taking the inverse transform

$$F_{xy}^{-1}\left[T(\underline{h_x E})\right] = h_x\{(1-h_x)\, g_z * h_x E\} + h_x g_z' * h_x\{(1-h_x)\, g_z * h_x E\} . \tag{129}$$

Now the first term in (129) may be written as

$$h_x \int_{x'} [1-h(x-x')]\, g_z(x-x')\, h(x')\, E(x')\, dx'$$

$$= h(x) \int_{x}^{\infty} g_z(x-x')\, E(x')\, dx' \ , \qquad (130)$$

where, for convenience of notation, we omit the y' integration, i.e., the above actually is

$$h(x) \int_{y'} \int_{x}^{\infty} g_z(x-x',\ y-y')\, E(x',y')\, dx'\, dy' \ .$$

The second term in (129) is therefore

$$\int_{x'} h(x-x')\, g_z'(x-x')\, h(x') \int_{x'}^{\infty} g_z(x'-x")\, E(x")\, dx"\, dx'$$

$$= \int_{0}^{x} g_z'(x-x') \int_{x'}^{\infty} g_z(x'-x")\, E(x")\, dx"\, dx' \ . \qquad (131)$$

Hence (129) may be written as,

$$F_{xy}^{-1}[T(\underline{hE})] = h(x) \left[\int_{x}^{\infty} g_z(x-x')\, E(x')\, dx' \right.$$

$$\left. + \int_{0}^{x} g_z'(x-x') \int_{x'}^{\infty} g_z(x'-x")\, E(x")\, dx"\, dx' \right] \ . \qquad (132)$$

Referring now to (128), we have

$$h_x E(x) = h_x E_{osz}(x) + \int_{0}^{x} g_z'(x-x')\, E_{osz}(x')\, dx' \ . \qquad (133)$$

Let us put

$$H_0(x) = E_{osz}(x) \ , \qquad (134)$$

then the first two terms of (103) may be written as,

$$h(x)\left[(H_0(x) + \int_0^x g_z'(x-x')\, H_0(x')\, dx')\right.$$

$$\left. + (H_1(x) + \int_0^x g_z'(x-x')\, H_1(x')\, dx')\right] \qquad (135)$$

where

$$H_1(x) = \int_x^\infty g_z(x-x')\left[H_0(x') + \int_0^{x'} g_z'\,(x'-x")\, H_0(x")\, dx"\right] dx'. \qquad (136)$$

Generalizing (136), we see that the series (103) is given by

$$h_x E_z{}^{z+}(x) = h(x) \sum_{n=0}^{\infty} \left[H_n(x) + \int_0^x g_z'\,(x-x')H_n(x)dx'\right] \qquad (137)$$

where

$$H_0(x) = E_{osz}(x)$$

$$H_n(x) = \int_x^\infty g_z(x-x')\left[H_{n-1}(x') + \int_0^x g_z'\,(x'-x")H_{n-1}(x")dx"\right]dx' \ .$$

If we assume, as discussed in the previous section, that

$$g_z = \frac{jk(\Delta_2-\Delta_1)}{2\pi} F_2(\rho)\, \frac{e^{-jk\rho}}{\rho}$$

$$g_z' = \frac{jk(\Delta_2-\Delta_1)}{2\pi} F_1(\rho)\, \frac{e^{-jk\rho}}{\rho} \ ,$$

and also assume as excitation a vertical dipole at $(-x_0,0)$, then

$$E_{osz} = \frac{I\Delta\ell k^2}{j\omega\varepsilon_0(2\pi)} F_2(\rho_0)\, \frac{e^{-jk\rho_0}}{\rho_0} \ .$$

In this case we have

$$H_0 = \frac{I\Delta\ell k^2}{j\omega\varepsilon_0(2\pi)} F_2(\rho_0) \frac{e^{-jk\rho_0}}{\rho_0} = \frac{I\Delta\ell k^2}{j\omega\varepsilon_0(2\pi)} P_0 \text{ (say);}$$

$$H_1 = \frac{I\Delta\ell k^2}{j\omega\varepsilon_0(2\pi)} \frac{jk(\Delta_2-\Delta_1)}{2\pi} \int_x^\infty F_2[\rho(x-x')] \frac{e^{-jk\rho(x-x')}}{\rho(x-x')}$$

$$\cdot\left[F_2[\rho(x_0+x')] \frac{e^{-jk\rho(x_0+x_1)}}{\rho(x_0+x')} + \frac{jk(\Delta_2-\Delta_1)}{2\pi} \int_0^x F_1[\rho(x'-x")]\right.$$

$$\left.\cdot F_2[\rho(x_0+x")] \frac{e^{-jk[\rho(x'-x") + \rho(x_0+x")]}}{\rho(x'-x")\rho(x_0+x")} dx"\right]dx' \qquad (138)$$

$$= \frac{I\Delta\ell k^2}{j\omega\varepsilon_0(2\pi)} \left[\frac{jk(\Delta_2-\Delta_1)}{2\pi}\right] \int_x^\infty F_2[\rho(x-x')] \frac{e^{-jk\rho(x-x')}}{\rho(x-x')} \left[P_0(x')\right.$$

$$\left. + \frac{jk(\Delta_2-\Delta_1)}{2\pi} \int_0^x F_1[\rho(x'-x")]P_0(x") \frac{e^{-jk\rho(x'-x")}}{\rho(x'-x")} dx"\right]dx'$$

$$= \frac{I\Delta\ell k^2}{j\omega\varepsilon_0(2\pi)} \left[\frac{jk(\Delta_2-\Delta_1)}{2\pi}\right] P_1 .$$

Then, in general

$$H_n = \frac{I\Delta\ell k^2}{j\omega\varepsilon_0(2\pi)} \left[\frac{jk(\Delta_2-\Delta_1)}{2\pi}\right]^n P_n , \qquad (139)$$

where

$$P_n = \int_x^\infty F_2[\rho(x-x')] \frac{e^{-jk\rho(x-x')}}{\rho(x-x')} \left[P_{n-1} + \frac{jk(\Delta_2-\Delta_1)}{2\pi} \int_0^x\right.$$

$$\left. F_1[\rho(x'-x")] P_{n-1}(x") \frac{e^{jk\rho(x'-x")}}{\rho(x'-x")} dx"\right] dx' . \qquad (140)$$

Therefore (137) becomes

$$h_x E_z^{z+} = h(x) \frac{I \Delta \ell k^2}{j\omega\varepsilon_0(2\pi)} \sum_{n=0}^{\infty} \left[\frac{jk(\Delta_2-\Delta_1)}{2\pi}\right]^n \left[P_n + \frac{jk(\Delta_2-\Delta_1)}{2\pi} \int_0^x F_1[\rho(x-x')] \frac{e^{-jk\rho(x-x')}}{\rho(x-x')} P_n(x')\, dx'\right] , \quad (141)$$

with P_n, $n > 1$, given by (140) and

$$P_0 = F_2(\rho_0) \frac{e^{-jk\rho_0}}{\rho_0} ,$$

$$\rho_0 = ((x+x_0)^2 + y^2)^{1/2} .$$

3.5 The Electric Field for the Case of two Simple Discontinuities

We now consider the problem of determining the electric surface field for the surface illustrated in Figure 7.

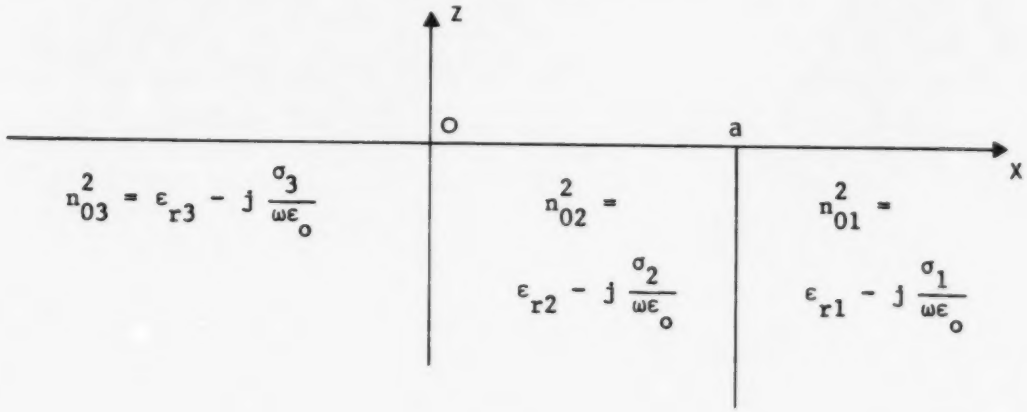


Figure 7 - Surface with two discontinuties

Referring to (93), it is obvious for this case and under the same assumptions as used in (93), that the surface field $\bar{E}^{z+}$ now obeys the equation,

$$2u\,\underline{\bar{E}_s^{z-}}\,e^{|z-|u} = (uI + \underline{A_3})(\underline{1 - h(x)})\underline{E^{z+}}$$

$$+ (uI + \underline{A_2})[\underline{h(x) - h(x-a)}]\underline{\bar{E}^{z+}} \qquad (142)$$

$$+ (uI + \underline{A_1})\,\underline{h(x-a)E^{z+}},$$

where $\underline{A_1}$, $\underline{A_2}$, $\underline{\bar{E}_s^{z-}}$ and $\underline{\bar{E}^{z+}}$, are as defined in (93) and

$$\underline{A_3} = \begin{bmatrix} jk\,n_{03}^2\Delta_3 & 0 & jk_x\,\dfrac{n_{03}^2-1}{n_{03}^2} \\ 0 & jk\,n_{03}^2\Delta_3 & jk_y\,\dfrac{n_{03}^2-1}{n_{03}^2} \\ 0 & 0 & jk\,\Delta_3 \end{bmatrix},$$

$$\Delta_3 = \frac{n_{03}^2 - 1}{n_{03}^2}\ .$$

For convenience, we shall only consider the vertical field E_z^{z+}, then from (142)

$$2u\,E_{sz}^{z-}\,e^{|z-|u} = (u + jk\Delta_3)[\underline{1 - h(x)}]\underline{E_z^{z+}}$$

$$+ (u + jk\Delta_2)[\underline{h(x) - h(x-a)}]\underline{E_z^{z+}} \qquad (143)$$

$$+ (u + jk\Delta_1)[\underline{h(x-a)}]\underline{E_z^{z+}}.$$

Equation (143) is conveniently written in the form

$$\frac{2u\,\underline{E_{sz}^{z+}}\,e^{|z-|u}}{u + jk\,\Delta_3} = \underline{E_z^{z+}} - \frac{jk(\Delta_3-\Delta_2)}{u + jk\Delta_3}\,\underline{[h(x) - h(x-a)]E_z^{z+}}$$

$$- \frac{jk(\Delta_3-\Delta_1)}{u + jk\Delta_3}\,\underline{h(x-a)E_z^{z+}} \qquad (144)$$

or, for convenience of notation,

$$\underline{E}_{osz} = \underline{E}_z^{z+} - jk(\Delta_3-\Delta_2)\,\underline{g}\,[h(x) - h(x-a)]E_z^{z+}$$

$$- jk(\Delta_3-\Delta_1)\,\underline{g}\,h(x-a)E_z^{z+}$$

where

$$\underline{E}_{osz} = \frac{2u\,E_{sz}^{z-}\,e^{|z-|u}}{u + jk\Delta_3}$$

$$\underline{g} = \frac{1}{u + jk\Delta_3} \quad .$$

Now we write

$$\underline{E_{osz}} = \underline{(1 - h_x)E_{osz}} + \underline{h_{xa}E_{osz}} + \underline{h_a E_{osz}}$$

$$\underline{E_z^{z+}} = \underline{(1-h_x)E_z^{z+}} + \underline{h_{xa}E_z^{z+}} + \underline{h_a E_z^{z+}} \qquad (145)$$

$$\underline{g} = \underline{(1-h_x)g} + \underline{h_{xa}g} + \underline{h_a g}$$

where again, for convenience of notation,

$$h_x = h(x)$$

$$h_{xa} = h(x) - h(x-a)$$

$$h_a = h(x-a) \quad .$$

For the class of problems we are interested in at the moment we have $\Delta_3 = \Delta_1$ or (144) is,

$$\underline{E_{osz}} = \underline{E_z^{z+}} - jk\,(\Delta_3 - \Delta_2)\,\underline{g}\,\underline{h_{xa}E_z^{z+}} \quad . \qquad (146)$$

Using our decompositions (145) gives

$$\underline{(1 - h_x)E_{0sz}} + \underline{h_{xa}}\,\underline{E_{0sz}} + \underline{h_z E_{0sz}}$$

$$= \underline{(1 - h_x)E_z^{z+}} + \underline{h_{xa}E_z^{z+}} + \underline{h_z E_z^{z+}}$$

$$- jk(\Delta_3-\Delta_2)(\underline{(1 - h_x)g} + \underline{h_{xa}g} + \underline{h_a g})\,\underline{h_{xa}E_z^{z+}} \quad . \qquad (147)$$

Comparing terms with like support, (147) implies

$$\underline{h_{xa}E_{0sz}} = \underline{h_{xa}E_x^{z+}} - jk(\Delta_3 - \Delta_2)[\underline{h_{xa}(h_{xa}g * h_{xa}E_z^{z+})}]$$

$$- jk(\Delta_3-\Delta_2)[\underline{h_{xa}\{((1-h_x)g + h_ag) * h_{xa}E_z^{z+}\}}]$$

$$\underline{(1-h_x)E_{0sz}} = \underline{(1-h_x)E_z^{z+}} - jk(\Delta_3-\Delta_2)[\underline{(1-h_x)\{((1-h_x)g + h_{xa}g + h_ag)}$$

$$\underline{* h_{xa}E_z^{z+}\}}]$$

$$\underline{h_aE_{0sz}} = \underline{h_aE_z^{z+}} - jk(\Delta_3-\Delta_2)[\underline{h_a\{((1-h_x)g + h_{xa}g + h_ag) * h_{xa}E_z^{z+}\}}]. \quad (148)$$

Again from support arguments, and using the identities,

$$h_{xa}(h_{xa}g * h_{xa}E_z^{z+}) = [1 - (1-h_{xa})](h_{xa}g * h_{xa}E_z^{z+})$$

$$= (h_{xa}g * h_{xa}E_z^{z+}) - (1-h_{xa})(h_{xa}g * h_{xa}E_z^{z+})$$

$$= (h_{xa}g * h_{xa}E_z^{z+}) - h_a(h_{xa}g * h_{xa}E_z^{z+}),$$

(148) may be written as,

$$\underline{h_{xa}E_{0sz}} = \underline{h_{xa}E_z^{z+}} - jk(\Delta_3-\Delta_2)\ \underline{h_{xa}g}\ \underline{h_{xa}E_z^{z+}}$$

$$- jk(\Delta_3-\Delta_2)[\underline{h_{xa}\{(1-h_x)g * h_{xa}E_z^{z+}\} - h_a\{h_{xa}g * h_{xa}E_z^{z+}\}}]$$

$$\underline{(1-h_x)E_{0sz}} = \underline{(1-h_x)E_z^{z+}} - jk(\Delta_3-\Delta_2)[\underline{(1-h_x)\{(1-h_x)g * h_{xa}E_z^{z+}\}}] \quad (149)$$

$$\underline{h_aE_{0sz}} = \underline{h_aE_z^{z+}} - jk(\Delta_3-\Delta_2)[\underline{h_a\{h_xg * h_{xa}E_x^{z+}\}}] \ .$$

The symbol * in equations (148) and (149) denotes two-dimensional (xy) convolution.

We consider the first of equations (149), which may be written in the form

$$\underline{h_{xa}E_z^{z+}} - \frac{jk(\Delta_3-\Delta_2)}{1-jk(\Delta_3-\Delta_2)\underline{h_{xa}g}}\left[\underline{h_{xa}\{(1-h_x)g * h_{xa}E_z^{z+}\}}\right.$$

$$\left. - \underline{h_a\{h_{xa}g * h_{xa}E_z^{z+}\}}\right]$$

$$= \frac{\underline{h_{xa}E_{0sz}}}{1-jk(\Delta_3-\Delta_2)\underline{h_{xa}g}}. \qquad (150)$$

As in section 3.2, (150) may be seen to be in the form

$$(I-T)\ \underline{h_{xa}E_z^{z+}} = \underline{E} \qquad (151)$$

where I is the identity operator and T is the linear operator

$$T = \frac{jk(\Delta_3-\Delta_2)}{1-jk(\Delta_3-\Delta_2)\underline{h_{xa}g}}\left[\underline{h_{xa}\{(1-h_x)g * (\ )\}} - \underline{h_a\{h_{xa}g * (\ )\}}\right],$$

and

$$\underline{E} = \frac{\underline{h_{xa}E_{0sz}}}{1-jk(\Delta_3-\Delta_2)\underline{h_{xa}g}}\ .$$

A formal solution of (151) is

$$\underline{h_{xa}E_z^{z+}} = \sum_n T^n\,[\underline{E}]. \qquad (152)$$

In particular the first order solution (n = 0) is,

$$\underline{h_{xa}E_z^{z+}} = \underline{h_{xa}E}. \qquad (153)$$

Now

$$\underline{E} = \frac{\underline{h_{xa}E_{0sz}}}{1-jk(\Delta_3-\Delta_2)\underline{h_{xa}g}}$$

$$= \left[1 + \frac{jk(\Delta_3-\Delta_2)\underline{h_{xa}g}}{1-jk(\Delta_3-\Delta_2)\underline{h_{xa}g}}\right]\ \underline{h_{xa}E_{0sz}}.$$

In the spirit of (106) we put

$$\frac{jk(\Delta_3-\Delta_2)\underline{h_{xa}g}}{1-jk(\Delta_3-\Delta_2)\underline{h_{xa}g}} \simeq jk(\Delta_3-\Delta_2)\underline{h_{xa}F_x^{-1}}\left\{\frac{\underline{g}}{1-jk(\Delta_3-\Delta_2)\underline{g}}\right\}$$

$$= jk(\Delta_3-\Delta_2)\,\underline{h_{xa}g'}\ ,$$

where

$$\underline{g'} = \frac{\underline{g}}{1-jk(\Delta_3-\Delta_2)\underline{g}} = \frac{\dfrac{1}{u+jk\Delta_3}}{1-\dfrac{jk(\Delta_3-\Delta_2)}{u+jk\Delta_3}} = \frac{1}{u+jk\Delta_2}\ .$$

Therefore from (153)

$$\underline{h_{xa}E_z^{z+}} \simeq [1+jk(\Delta_3-\Delta_2)\underline{h_{xa}g'}]\underline{h_{xa}E_{0sz}}$$

$$= \underline{h_{xa}E_{0sz}} + jk(\Delta_3-\Delta_2)\underline{h_{xa}g'}\ \underline{h_{xa}E_{0sz}}, \qquad (154)$$

to first order.

From (149), also to first order,

$$\underline{(1-h_x)E_z^{z+}} = \underline{(1-h_x)E_{0sz}} + jk(\Delta_3-\Delta_2)[\underline{(1-h_x)\{(1-h_x)g}$$

$$\underline{* (h_{xa}E_{0sz} + jk(\Delta_3-\Delta_2)h_{xa}g' * h_{xa}E_{0sz})\}]}, \qquad (155)$$

and,

$$\underline{h_aE_z^{z+}} = \underline{h_aE_{0sz}} + jk(\Delta_3-\Delta_2)[\underline{h_a\{h_xg * (h_{xa}E_{0sz}}$$

$$\underline{+ jk(\Delta_3-\Delta_2)h_{xa}g' * h_{xa}E_{0sz})\}]}. \qquad (156)$$

We shall show in the following that (156) agrees with Bremmer (1954), (after stationary phase integrations) for the case of a vertical dipole source.

Let us now assume a vertical dipole at coordinates $(-x_0, 0, 0)$ as the source, then from (120)

$$E_{0sz} \simeq \frac{I\ \Delta\ \ell\ k^2}{j\omega\varepsilon_0(2\pi)} \frac{F_3(\rho_0)\, e^{-jk\rho_0}}{\rho_0} \tag{157}$$

where

$$\rho_0 = ((x + x_0)^2 + y^2)^{1/2}$$

$$F_3(\rho_0) = 1 - j\sqrt{\pi p_3}\, e^{-p_3}\, \text{erfc}\,(j\sqrt{p_3})$$

$$p_3 = -jk\frac{\rho_0}{2}\Delta_3^2.$$

Also

$$g \simeq \frac{1}{2\pi} F_3(\rho) \frac{e^{-jk\rho}}{\rho} \tag{158}$$

$$\rho = (x^2 + y^2)^{1/2}$$

and $$g' \simeq \frac{1}{2\pi} F_2(\rho) \frac{e^{-jk\rho}}{\rho} \tag{159}$$

$$F_2(\rho) = 1 - j\sqrt{\pi p_2}\, e^{-p_2}\, \text{erfc}\,(j\sqrt{p_2})$$

$$p_2 = -jk\frac{\rho}{2}\Delta_2^2.$$

Using (157), (158) and (159), (154), (155) and (156) may be written in the form of convolutions as,

$$h_{xa}E_z^{z+} = h_{xa}\left[\frac{I\ \Delta\ \ell\ k^2}{j\omega\varepsilon_0(2\pi)}\left\{F_3(\rho_0)\frac{e^{-jk\rho_0}}{\rho_0} + \frac{jk(\Delta_3-\Delta_2)}{2\pi}\int_{y'}\int_0^x F_2(\rho_c)F_3(\rho'_0)\frac{e^{-jk[\rho_c+\rho'_0]}}{\rho_c\rho'_0}\,dx'dy'\right\}\right], \tag{160}$$

$$\rho_c = ((x-x')^2 + (y-y')^2)^{1/2},$$

$$\rho_0' = ((x_0+x')^2 + y'^2)^{1/2}$$

$$(1-h_x)E_z^{z+} = (1-h_x)\left[\frac{I\Delta\ell k^2}{j\omega\varepsilon_0(2\pi)}\left\{F_3(\rho_0)\frac{e^{-jk\rho_0}}{\rho_0} + \frac{jk(\Delta_3-\Delta_2)}{2\pi}\right.\right.$$

$$\cdot\int_{-\infty}^{\infty}\int_{0}^{a} F_3(\rho_c)\frac{e^{-jk\rho_c}}{\rho_c}\left(F_3(\rho_0')\frac{e^{-jk\rho_0'}}{\rho_0'} + \frac{jk(\Delta_3-\Delta_2)}{2\pi}\right.$$

$$\left.\left.\cdot\int_{-\infty}^{\infty}\int_{0}^{x'} F_2(\rho_a)F_3(\rho_0)\frac{e^{-jk(\rho_a+\rho_0)}}{\rho_a\rho_b}\,dx''dy'')\,dx'dy'\right\}\right], \tag{161}$$

$$\rho_c = ((x-x')+(y-y')^2)^{1/2}$$

$$\rho_a = ((x'-x'')^2 + (y'-y'')^2)^{1/2}$$

$$\rho_b = ((x_0+x'')^2 + y''^2)^{1/2}$$

$$h_a E_z^{z+} = h_a\left[\frac{I\Delta\ell k^2}{j\omega\varepsilon_0(2\pi)}\left\{F_3(\rho_0)\frac{e^{-jk\rho_0}}{\rho_0} + \frac{jk(\Delta_3-\Delta_2)}{2\pi}\cdot\right.\right.$$

$$\int_{-\infty}^{\infty}\int_{0}^{a} F_3(\rho_c)\frac{e^{-jk\rho_c}}{\rho_c}\left(F_3(\rho_0')\frac{e^{-jk\rho_0'}}{\rho_0'} + \frac{jk(\Delta_3-\Delta_2)}{2\pi}\cdot\right.$$

$$\left.\left.\int_{-\infty}^{\infty}\int_{0}^{x'} F_2(\rho_a)F_3(\rho_b)\frac{e^{-jk(\rho_a+\rho_b)}}{\rho_a\rho_b}\,dx''\,dy'')\,dx'\,dy'\right\}\right], \tag{162}$$

$$\rho_c = ((x-x')+(y-y')^2)^{1/2}$$

$$\rho_a = ((x'-x'')^2 + (y'-y'')^2)^{1/2}$$

$$\rho_b = ((x_0+x'')^2 + y''^2)^{1/2}$$

If we use the stationary phase integration as in (125), then (160), (161) and (162) are,

$$h_{xa} E_z^{z+} = h_{xa} \frac{I\Delta\ell k^2}{j\omega\varepsilon_0(2\pi)} \frac{e^{-jk\rho_0}}{\rho_0} \left[F_3(\rho_0) \right.$$

$$\left. + \left(\frac{jk}{2\pi}\right)^{1/2} (\Delta_3 - \Delta_2) \sqrt{\rho_0} \int_{x_0 \sec\theta}^{\rho_0} \frac{F_2(\rho_0 - \rho') F_3(\rho_1)}{\sqrt{\rho_0 - \rho'}\ \sqrt{\rho'}}\, d\rho' \right] \qquad (163)$$

$$(1-h_x) E_z^{z+} = (1-h_x) \frac{I\Delta\ell k^2}{j\omega\varepsilon_0(2\pi)} \left[F_3(\rho_0) \frac{e^{-jk\rho_0}}{\rho_0} + \frac{jk(\Delta_3 - \Delta_2)}{2\pi} \cdot \right.$$

$$\cdot \int_{-\infty}^{\infty} \int_0^a F_3(\rho_c) \frac{e^{-jk(\rho_c + \rho_0')}}{\rho_c \rho_0'} \left\{F_3(\rho_0') + (\Delta_3 - \Delta_2) \left(\frac{jk\rho_0'}{2\pi}\right)^{1/2} \right.$$

$$\left. \cdot \int_{x_0 \sec\theta}^{\rho_0'} \frac{F_2(\rho_0' - \rho') F_3(\rho')}{\sqrt{\rho_0' - \rho'}\ \sqrt{\rho'}}\, d\rho'\, dx'\, dy' \right] \qquad (164)$$

$$h_a E_z^{z+} = h_a \frac{I\Delta\ell k^2}{j\omega\varepsilon_0(2\pi)} \left[F_3(\rho_o) \frac{e^{-jk\rho_0}}{\rho_0} + \frac{jk(\Delta_3 - \Delta_2)}{2\pi} \cdot \right.$$

$$\int_{-\infty}^{\infty} \int_0^a F_3(\rho_c) \frac{e^{-jk(\rho_c + \rho_0')}}{\rho_c \rho_0'} \left\{F_3(\rho_0') + (\Delta_3 - \Delta_2) \left(\frac{jk\rho_0'}{2\pi}\right)^{1/2} \right.$$

$$\left. \int_{x_0 \sec\theta}^{\rho_0'} \frac{F_2(\rho_0' - \rho') F_3(\rho')}{((\rho_0' - \rho')\rho')^{1/2}}\, d\rho'\, dx'\, dy' \right] . \qquad (165)$$

The angle θ is defined in Figure 8. Also we note that a second stationary phase integration may be performed in (165).

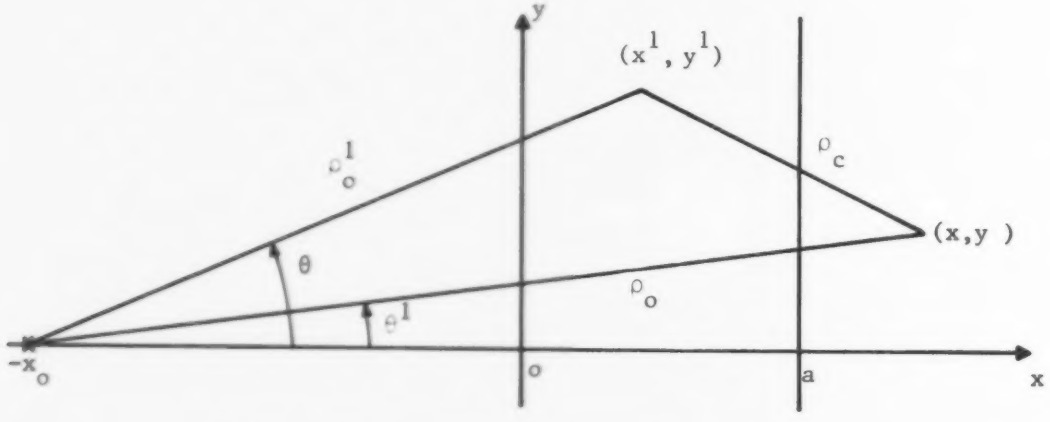


Figure 8. Variables in Equation (165).

Refer to Figure 8 which illustrates the quantities in the double integral of (165). Using arguments similiar to these leading up to (125), we find that the double integral of (165)

$$\simeq (\Delta_3-\Delta_2)\ (\tfrac{jk}{2\pi})^{1/2}\ \frac{e^{-jk\rho_0}}{\sqrt{\rho_0}} \int_{x_0\sec\theta'}^{[x_0+a]} \frac{F_3(\rho_0-\rho_0')}{\sqrt{\rho_0-\rho_0'}\ \sqrt{\rho_0'}} \{F_3(\rho_0')$$

$$+ (\Delta_3-\Delta_2)\ (\tfrac{jk\rho_0'}{2\pi})^{1/2} \int_{x_0\sec\theta'}^{\rho_0} \frac{F_2(\rho_0' - '\rho')}{\sqrt{\rho_0' - \rho'}} \frac{F_3(\rho')}{\sqrt{\rho'}}\, d\rho'\}d\rho_0'$$

Therefore (165) may be expressed as,

$$h_a E^{z+} = h_a \frac{I\ \Delta\ \ell\ k^2}{j\omega\epsilon_0(2\pi)} \frac{e^{-jk\rho_0}}{\rho_0} [F_3\ (\rho_0) + (\Delta_3-\Delta_2)\ (\tfrac{jk\rho_0}{2\pi})^{1/2}.$$

$$\int_c^d \frac{F_3(\rho\ -\rho\ ')}{((\rho_0-\rho_0')\rho_0')^{1/2}} \{F_3(\rho_0') + (\Delta_3-\Delta_2)\ (\tfrac{jk\rho_0'}{2\pi})^{1/2} \int_c^{\rho_o'}$$

$$\cdot\ \frac{F_2(\rho_0' - \rho')F_3(\rho')}{((\rho_0'- \rho)\rho')^{1/2}}\, d\rho'\}d\rho_0'\ ]. \qquad (166)$$

where $c = x_0 \sec\theta'$, $d = (x_0 + a)\sec\theta'$:

Higher order terms in (152) may also be derived as in Section (3.4). It may readily be seen that (166) agrees with Bremmer (1954) for $u_1=u_3$ in his notation.

3.6 The Electric Field for The Case of a Discontinuity Involving a Finite Region

Let us now consider the case illustrated in Figure 9.

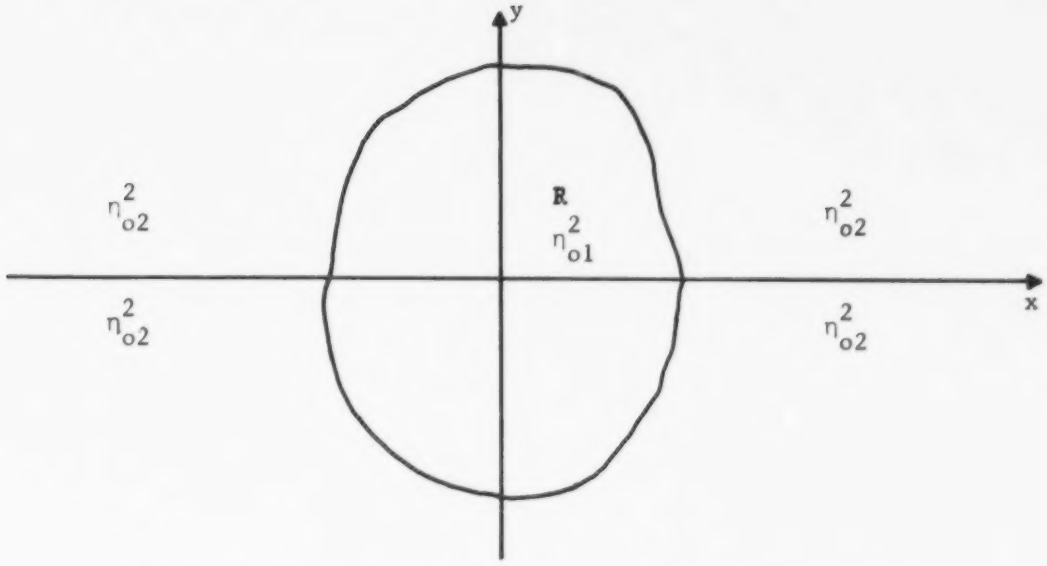


Figure 9 Surface Discontinuity in a Bounded Region R

Referring to the previous sections, i.e. equations (93) and (142), and under the same assumptions previously employed, the electric field $\bar{E}^{z+}$ may now be assumed to satisfy the equation

$$2u\underline{\bar{E}}_s^{z-} e^{|z-|u} = (uI + \underline{A}_2)(1 - h_R)\underline{\bar{E}^{z+}} + (uI + \underline{A}_1)h_R\underline{\bar{E}^{z+}} \qquad (167)$$

where $\underline{A}_1$, $\underline{A}_2$, $\underline{\bar{E}}_s^{z-}$ and $\underline{\bar{E}}^{z+}$ may be taken to be defined as in (93) and the Heaviside function h_R is defined as the characteristic function of the interior (excluding the boundary) of the set R, i.e.

$$h_R(x,y) \quad \begin{matrix} 1 & (x,y) \in R \\ 0 & (x,y) \quad R \end{matrix} .$$

Again, for convenience, we shall only consider the vertical component of the field E_z^{z+}. Then we consider the component of (167),

$$2u\,\underline{E}_{sz}^{z-} e^{|z-|u} = (u + jk\Delta_2)(1 - h_R)\underline{E_z^{z+}} + (u + jk\Delta_1)h_R\underline{E_z^{z+}} \quad . \qquad (168)$$

Equation (168) is more conveniently written as

$$\frac{2u\ \underline{E}_{sz}^{z-}\ e^{|z-|u}}{u+jk\Delta_2} = \underline{E}_z^{z+} - \frac{jk(\Delta_2-\Delta_1)}{u+jk\Delta_2}\ h_R\underline{E}_z^{z+} \tag{169}$$

or

$$\underline{E}_{0sz} = \underline{E}_z^{z+} - jk(\Delta_2-\Delta_1)\ g\ h_R\underline{E}_z^{z+},$$

where $\underline{E}_{0sz}$ and g have obvious definitions.

As before we decompose $\underline{E}_{0sz}$ as,

$$\underline{E}_{0sz} = (1-h_R)\underline{E}_{0sz} + h_R\underline{E}_{0sz} \tag{170}$$

and rewrite (169) as,

$$(1-h_R)\underline{E}_{0sz} + h_R\underline{E}_{0sz} = (1-h_R)\underline{E}_z^{z+} + h_R\underline{E}_z^{z+}$$

$$- jk(\Delta_2-\Delta_1)[h_R\{g * h_R\underline{E}_z^{z+}\} + (1-h_R)\{g * h_R\underline{E}_z^{z+}\}].$$

This equation implies

$$h_R\underline{E}_{0sz} = [1-jk(\Delta_2-\Delta_1)\ g]h_R\underline{E}_z^{z+} + jk(\Delta_2-\Delta_1)(1-h_R)(g * h_R\underline{E}_z^{z+}) \tag{171}$$

$$(1-h_R)\underline{E}_{0sz} = (1-h_R)\underline{E}_z^{z+} - jk(\Delta_2-\Delta_1)[(1-h_R)\{g * h_R\underline{E}_z^{z+}\}] \tag{172}$$

where * again denotes two dimensional (x,y) convolution. We first consider (171), which may be written in the following form,

$$h_R\underline{E}_z^{z+} + jk\ \frac{\Delta_2-\Delta_1}{1-jk(\Delta_2-\Delta_1)\ g}\ (1-h_R)\{g * h_R\underline{E}_z^{z+}\} = \frac{h_R\underline{E}_{0sz}}{1-jk(\Delta_2-\Delta_1)\ g}. \tag{173}$$

Now (173) is of the form,

$$(I + T)h_R\underline{E}_z^{z+} = \underline{E} \tag{174}$$

where I is the identity operator and T is the linear operator

$$T(\ ) = \frac{jk(\Delta_2-\Delta_1)}{1-jk(\Delta_2-\Delta_1)\ g}\ (1-h_R)\{g * (\ )\}$$

and

$$\underline{E} = \frac{\underline{h_R E_{0sz}}}{1-jk(\Delta_2-\Delta_1)\ \underline{g}} .$$

The formal solution of (174) is,

$$\underline{h\ E_z^{z+}} = \sum_n (-1)^n T^n(\underline{E}) \tag{175}$$

In particular, the first order estimate is,

$$\underline{h_R E_z^{z+}} = \underline{h_R E} .$$

Now

$$\underline{E} = \frac{\underline{h_R E_{0sz}}}{1-jk(\Delta_2-\Delta_1)\ \underline{g}} = [1 + \frac{jk(\Delta_2-\Delta_1)\ \underline{g}}{1-jk(\Delta_2-\Delta_1)\ \underline{g}}]\underline{h_R E_{0sz}}, \text{ and,}$$

$$\frac{jk(\Delta_2-\Delta_1)\ \underline{g}}{1-jk(\Delta_2-\Delta_1)\ \underline{g}} = jk(\Delta_2-\Delta_1)\ \underline{g}' ,$$

where

$$g' = \frac{\underline{g}}{1-jk(\Delta_2-\Delta_1)\ \underline{g}} = \frac{1}{u+jk\Delta_1} .$$

Therefore, to first order,

$$\underline{h_R E_z^{z+}} = \underline{h_R E_{0sz}} + jk(\Delta_2-\Delta_1)\underline{h_R\ \{g' * h_R E_{0sz}\}} . \tag{177}$$

Also, from (172) using (177),

$$\underline{(1-h_R)E_z^{z+}} = \underline{(1-h_R)E_{0sz}}$$

$$+ jk(\Delta_2-\Delta_1)[\underline{(1-h_R)\{g * h_R\{E_{0sz} + jk(\Delta_2-\Delta_1)\ g' * h_R E_{0sz}\}\}}.$$

If we now assume a vertical dipole at coordinates $(-x_0,0,0)$ as the source, then

$$E_{0sz} \simeq \frac{I\Delta\ell k^2}{j\omega\varepsilon_0(2\pi)} \frac{F_2(\rho_0)}{\rho_0} e^{-jk\rho_0} \tag{179}$$

where

$$\rho_0 = ((x+x_0)^2 + y^2)^{1/2}$$

$$F_2(\rho_0) = 1 - j\sqrt{\pi p_2}\, e^{-p_2}\, \mathrm{erfc}\,(j\sqrt{p_2})$$

$$p_2 = -jk\frac{\rho_0}{2}\Delta_2^2.$$

Also,

$$g \simeq \frac{1}{2\pi}\frac{F_2(\rho)}{\rho} e^{-jk\rho} \tag{180}$$

$$g' \simeq \frac{1}{2\pi}\frac{F_1(\rho)}{\rho} \quad -jk\rho, \tag{181}$$

where

$$\rho = (x^2 + y^2)^{1/2}$$

$$F_1(\rho) = 1 - j\sqrt{\pi p_1}\, e^{-p_1}\, \mathrm{erfc}\,(j\sqrt{p_1})$$

$$p_1 = -jk\frac{\Delta_1^2}{2}\rho.$$

Then, for the case of vertical dipole excitation, (177) and (178) may be written in the form of convolutions as

$$h_R E_z^{z+} = h_R \frac{I\Delta\ell k^2}{j\omega\varepsilon_0(2\pi)} \left[F_2(\rho)\frac{e^{-jk\rho_0}}{\rho_0} + \frac{jk(\Delta_2-\Delta_1)}{2\pi}\iint_R F_1(\rho_c)F_2(\rho_0')\frac{e^{-jk[\rho_c+\rho_0']}}{\rho_c\rho_0'}\,dx'dy'\right], \tag{182}$$

$$\rho_c = ((x-x')^2 + (y-y')^2)^{1/2}$$

$$\rho_o' = ((x'+x_0)^2 + y'^2)^{1/2}$$

$$(1-h_R)E^{z+} = (1-h_R)\frac{I\Delta\ell k^2}{j\omega\varepsilon_0(2\pi)}\left[F_2(\rho_0)\frac{e^{-jk\rho_0}}{\rho_0}\right.$$

$$+\frac{jk(\Delta_2-\Delta_1)}{2\pi}\iint_R F_2(\rho_c)\frac{e^{-jk\rho_c}}{\rho_c}\left\{F_2(\rho_0')\frac{e^{-jk\rho_0'}}{\rho_0'}\right.$$

$$\left.\left.+\frac{jk(\Delta_2-\Delta_1)}{2\pi}\iint_R F_1(\rho_a)F_2(\rho_0'')\frac{e^{-jk(\rho_a+\rho_0'')}}{\rho_a\rho_0''}dx''dy''\right\}dx'dy'\right] \tag{183}$$

where

$$\rho_0 = ((x+x_0)^2 + y^2)^{1/2}$$

$$\rho_c = ((x-x')^2 + (y-y')^2)^{1/2}$$

$$\rho_a = ((x'-x'')^2 + (y'+y'')^2)^{1/2}$$

$$\rho_0'' = ((x_0+x'')^2 + y''^2)^{1/2}$$

$$\rho_0' = ((x_0+x')^2 + y'^2)^{1/2}.$$

In this case it is not immediately obvious that stationary phase integration may be used in (182). We will discuss this further in the next chapter.

4. ASYMPTOTIC EXPANSIONS AND RADAR CROSS SECTIONS

In this Chapter we shall give a series of results based on the solutions of Chapter 3. We shall concentrate on those topics which we consider to be most useful from the point of view of application, in particular, the development of models for the remote sensing of sea ice with surface based ground wave radars. Further development of the theory, particulary section 3.4 will be deferred until a later time.

4.1 The Backscattered Field of a Simple Discontinuity, the Monostatic Case

Refer now to the term involving the double integral in (126), i.e.,

$$\left[(1 - h_x)E_z^{z+}\right]_b = (1 - h_x)\,\frac{I\,\Delta\,\ell k^2}{j\,\omega\,\varepsilon_o(2\pi)} \cdot \frac{jk(\Delta_2 - \Delta_1)}{2\pi} \cdot$$

$$\int\int_{-\infty_o}^{\infty\infty} F_2(\rho_c)\,\frac{e^{-jk(\rho_c + \rho_o')}}{\rho_c \rho_o'}\,\{F_2(\rho_o')$$

$$+ (\Delta_2 - \Delta_1)\,\left(\frac{jk\rho_o'}{2\pi}\right)^{1/2} \int_{x_o}^{\rho_o'} \sec\phi$$

$$\frac{F_1(\rho_o' - \rho')\,F_2(\rho')}{((\rho_o' - \rho')\,\rho')^{1/2}}\,d\rho\}\,dx'\,dy', \qquad (184)$$

where

$$\rho_c = (x - x')^2 + (y - y')^2$$

$$\rho_o' = (x_o + x')^2 + y'^2 \quad .$$

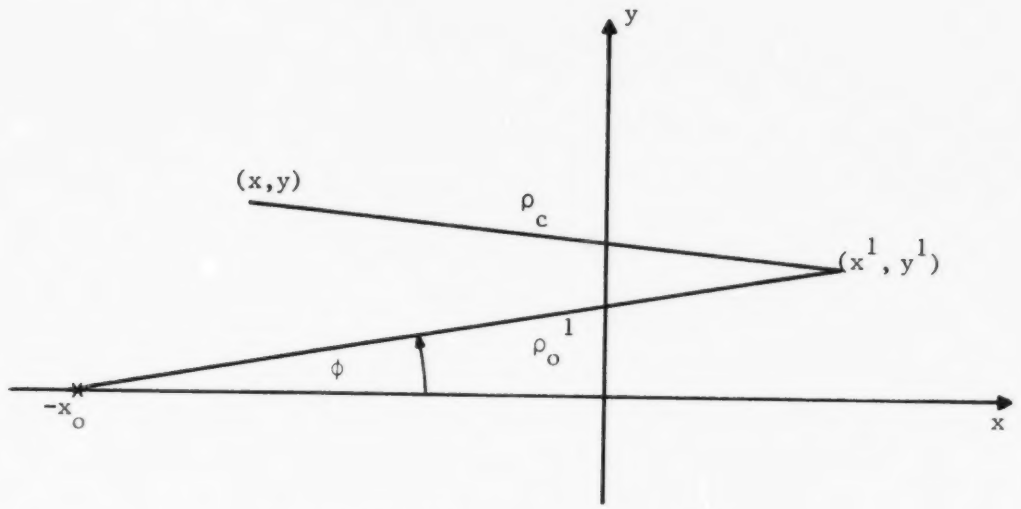


Figure 10 Illustration for (184)

If we change the variable of integration in (184) to (ρ_o', ϕ), then the double integral becomes

$$\int_{-\pi/2}^{\pi/2} \int_{x_o \sec\phi}^{\infty} F_2(\rho_c) \frac{e^{-jk(\rho_c + \rho_o')}}{\rho_c} \{F_2(\rho_o')$$

$$+ (\Delta_2 - \Delta_1) \left(\frac{jk\rho_o'}{2\pi}\right)^{1/2} \int_{x_o \sec\phi}^{\rho_o'} \frac{F_1(\rho_o' - \rho')F_2(\rho')}{((\rho_o' - \rho')\,\rho')^{1/2}} d\rho'\} d\rho_o' d\phi. \quad (185)$$

We consider the case $x = -x_o$, $y = 0$, i.e. the expression for $[E_z^{z+}]_b \mid_{(-x_o,\, 0)}$. Then the inner integral in (185) is, since $\rho_c = \rho_o'$,

$$\int_{x_o \sec\phi}^{\infty} e^{-2jk\rho_o'} \{\frac{F_2^2(\rho_o')}{\rho_o'}$$

$$+ (\Delta_2 - \Delta_1) \frac{F_2(\rho_o')}{\sqrt{\rho_o'}} \left(\frac{jk}{2\pi}\right)^{1/2} \int_{x_o \sec\phi}^{\rho_o'} \frac{F_1(\rho_o' - \rho')F_2(\rho')}{((\rho_o' - \rho')\,\rho')^{1/2}} d\rho'\} d\rho_o'. \quad (186)$$

For convenience, we rewrite (186) as,

$$\int_a^{\infty} e^{-2jk\rho_o'} G(\rho_o')\, d\rho_o', \quad (187)$$

where $G(\rho_o')$ is the bracketed quantity and

$$a = x_o \sec \phi \quad .$$

Now, it may be shown (see also Bremmer, 1954) that near the boundary $G(\rho_o')$ may be expanded as,

$$G(\rho_o') \simeq [\frac{F_2^2(a)}{a}]\{1 + 2b(\rho_o' - a)^{1/2} - jb\sqrt{\pi c_1}\,(\rho_o' - a) - \frac{4}{3}\,bc_1\,(\rho_o' - a)^{3/2} + jbc_1\sqrt{\pi c_1}\,\frac{(\rho_o' - a)^2}{2!} + ..\}$$
$$+ [\frac{F_2^2(a)}{a}]'(\rho_o' - a)\{1 + 2b(\rho_o' - a)^{1/2} - jb\sqrt{\pi c_1}(\rho_o' - a) - ...\}$$
$$+ [\frac{F_2^2(a)}{a}]''\frac{(\rho_o' - a)^2}{2!}\{1 + 2b(\rho_o' - a)^{1/2} - ...\} \qquad (188)$$

where the primes indicate derivatives and

$$b = (\frac{jk}{2\pi})^{1/2}\,(\Delta_2 - \Delta_1)$$

$$c_1 = -jk\,\frac{\Delta_1^2}{2} \quad .$$

Term by term integration of (188) in (187) gives, (assuming k has a small loss term)

$$\int_a^\infty e^{-2jk\rho_o'}\,G(\rho_o')\,d\rho_o' \simeq e^{-2jka}\,[\frac{F_2^2(a)}{a}\,\{\frac{\Gamma(1)}{jk} + 2b\,\frac{\Gamma(3/2)}{(jk)^{3/2}} - jb\sqrt{\pi c_1}\,\frac{\Gamma(2)}{(jk)^2} - \frac{4}{3}\,bc_1\,\frac{\Gamma(5/2)}{(jk)^{5/2}} + j\,\frac{bc_1\sqrt{\pi c_1}\;\Gamma(3)}{2(jk)^3} + ...\}$$
$$+ [\frac{F_2^2(a)}{a}]'\,\{\frac{\Gamma(2)}{(jk)^2} + 2b\,\frac{\Gamma(5/2)}{(jk)^{5/2}} + ...\}$$

$$+ \frac{1}{2!}\left[\frac{F_2^2(a)}{a}\right]'' \left\{\frac{\Gamma(3)}{(jk)^3} + \dots \right\}\Big], \tag{189}$$

where Γ is the gamma function.

Consider the series

$$\frac{\Gamma(1)}{jk} + 2b\,\frac{\Gamma(3/2)}{(jk)^{3/2}} - jb\,\pi c_1\,\frac{\Gamma(2)}{(jk)^2} - \frac{4}{3}\,bc_1\,\frac{\Gamma(5/2)}{(jk)^{5/2}}$$

$$+ j\,\frac{bc_1\sqrt{\pi c_1}}{2}\,\frac{\Gamma(3)}{(jk)^3} + \dots$$

$$= \frac{1}{jk}\left\{1 + \frac{\Delta_2 - \Delta_1}{\sqrt{2}}\left[1 + \frac{\Delta_1}{(2)^{1/2}} + \frac{\Delta_1^2}{(2)} + \frac{\Delta_1^3}{(2)^{3/2}} + \dots\right\}\right.$$

$$= \frac{1}{jk}\left\{1 + \frac{\Delta_2 - \Delta_1}{\sqrt{2}}\sum_n \left[\frac{\Delta_1}{\sqrt{2}}\right]^n\right\}$$

$$= \frac{1}{jk}\left\{1 + \frac{\Delta_2 - \Delta_1}{\sqrt{2} - \Delta_1}\right\},$$

since, by assumption $|\Delta_1| < 1$. Proceeding in the same way with the other series appearing in (189), we can rewrite (189) in the form

$$\int_a^\infty e^{-2jk\rho_o'}\,G(\rho_o')\,d\rho_o' \simeq e^{-2jka}\left[\frac{F_2^2(a)}{jka}\left\{1 + \frac{\Delta_2 - \Delta_1}{\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2} - \Delta_1}\right)\right\}\right.$$

$$+ \frac{1}{(jk)^2}\left(\frac{F_2^2(a)}{a}\right)'\left\{1 + \frac{\Delta_2 - \Delta_1}{\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2} - \Delta_1} + \frac{2}{(\sqrt{2} - \Delta_1)^2}\right)\right\}$$

$$+ \frac{1}{(jk)^3}\left(\frac{F_2^2(a)}{a}\right)''\left\{1 + \frac{\Delta_2 - \Delta_1}{2}\left(\frac{2}{\sqrt{2} - \Delta_1} + \frac{2}{(\sqrt{2} - \Delta_1)^2}\right.\right.$$

$$\left.\left.\left. + \frac{2\sqrt{2}}{(\sqrt{2} - \Delta_1)^3}\right)\right\} + \dots \right] \tag{190}$$

Now,

$$\left(\frac{F_2^2(a)}{a}\right)' = 2\,\frac{F_2(a)}{\sqrt{a}}\left(\frac{F_2(a)}{\sqrt{a}}\right)'$$

and (see (117)),

$$\frac{F_2(a)}{\sqrt{a}} = \frac{1}{\sqrt{a}} - j\sqrt{\pi c_2}\, e^{-c_2 a}\,\mathrm{erfc}\,(j\sqrt{c_2 a}),\quad c_2 = -jk\,\frac{\Delta_2^2}{2},$$

hence,

$$\left(\frac{F_2(a)}{a}\right)' = -\frac{1}{2a^{3/2}} - c_2\,\frac{F_2(a)}{\sqrt{a}}.$$

Then since

$$\frac{F_2(a)}{a} \simeq \left\{\frac{1}{2c_2 a^{3/2}} + \frac{1.3}{(2c_2)^2 a^{5/2}} + \frac{1.3.5}{(2c_2)^3 a^{7/2}} + \ldots\right\}, \tag{191}$$

we have

$$\left(\frac{F_2^2(a)}{a}\right)' = 0\left(\frac{1}{a^4}\right),\quad a \to \infty,$$

while

$$\left(\frac{F_2^2(a)}{a}\right) = 0\left(\frac{1}{a^3}\right),\quad a \to \infty.$$

In general

$$\left(\frac{F_2^2(a)}{a}\right)^{(n)} = 0\left(\frac{1}{a^{3+n}}\right),\quad a \to \infty. \tag{192}$$

Referring to (190) and (192), it may be seen that (190) is in the form of an asymptotic expansion. In particular we shall consider it sufficient to write (190) as

$$\simeq e^{-2jka}\, M(\Delta_2 - \Delta_1,\ \Delta_1)\, \frac{F_2^2(a)}{jka} \tag{193}$$

where

$$M(\Delta_2 - \Delta_1,\ \Delta_1) = 1 + \frac{\Delta_2 - \Delta_1}{\sqrt{2} - \Delta_1} \ .$$

However, higher order terms in the asymptotic expansion are available if necessary.

Using then (193) and (186), (184) may be written as,

$$\left[(1 - h_x)E_z^{z+}\right]_b\Big|(-x_o,\ 0) \simeq \frac{I\Delta\ell k^2}{j\omega\varepsilon_o(2\pi)}\, \frac{(\Delta_2 - \Delta_1)}{2\pi}\, M \int_{-\pi/2}^{\pi/2} \frac{F_2^2(x_o \sec\phi)}{x_o \sec\phi}$$

$$\cdot\ e^{-2jkx_o \sec\phi}\, d\phi \ . \tag{194}$$

If we do a stationary phase integration on (194) with respect to ϕ, the result is,

$$\left[(1 - h_x)E_z^{z+}\right]_b\Big|(x_o,\ 0) \simeq \frac{I\Delta\ell k^2}{j\omega\varepsilon_o(2\pi)}\, \frac{(\Delta_2 - \Delta_1)}{2\pi}\, M\, \left(\frac{\lambda_o x_o}{2}\right)^{1/2} \cdot$$

$$\frac{F_2^2(x_o)}{x_o^2}\, e^{-j(2kx_o + \frac{\pi}{4})} \tag{195}$$

where λ_o is the freespace electrical wavelength.

4.2 The Backscattered Field for the Case of Two Simple Discontinuities, the Monostatic Case

Refer now to (164). We shall seek an expression for the quantity involving the double integral, i.e.,

$$\left[(1 - h_x)E_z^{z+}\right]_b = (1 - h_x)\, \frac{I\Delta\ell k^2}{j\omega\varepsilon_o(2\pi)}\, \frac{jk(\Delta_3 - \Delta_2)}{2\pi} \iint_{-\infty_o}^{\infty_a} F_3(\rho_c)$$

$$\cdot \frac{e^{-jk(\rho_c + \rho_o')}}{\rho_c \rho_o'} \left\{ F_3(\rho_o') + (\Delta_3 - \Delta_2)\left(\frac{jk\rho_o'}{2\pi}\right)^{1/2} \int_{x_o \sec\phi}^{\rho_o'} \cdot \frac{F_2(\rho_o' - \rho')}{(\rho_o' - \rho')^{1/2}} \frac{F_3(\rho')}{(\rho')^{1/2}} d\rho' \right\} dx'\, dy' \quad (196)$$

where

$$\rho_c = (x - x')^2 + (y - y')^2$$

$$\rho_o' = (x_o + x')^2 + y'^2 \ .$$

The various quantites are illustrated in Figure 11.

As in the previous section we change the variables of integration to (ρ_o', ϕ), then the double integral in (196) becomes

$$\int_{-\pi/2}^{\pi/2} \int_{p}^{q} F_3(\rho_c) \frac{e^{-jk(\rho_c + \rho_o')}}{\rho_c} \left\{ F_3(\rho_o') + (\Delta_3 - \Delta_2)\left(\frac{jk\rho_o'}{2\pi}\right)^{1/2} \int_{\rho}^{\rho_o'} \frac{F_2(\rho_o' - \rho')}{\rho_o' - \rho'} \frac{F_3(\rho')}{\rho'} d\rho' \right\} d\rho_o'\, d\phi. \quad (197)$$

where

$$p = x_o \sec\phi$$

$$q = (x_o + a) \sec\phi \ .$$

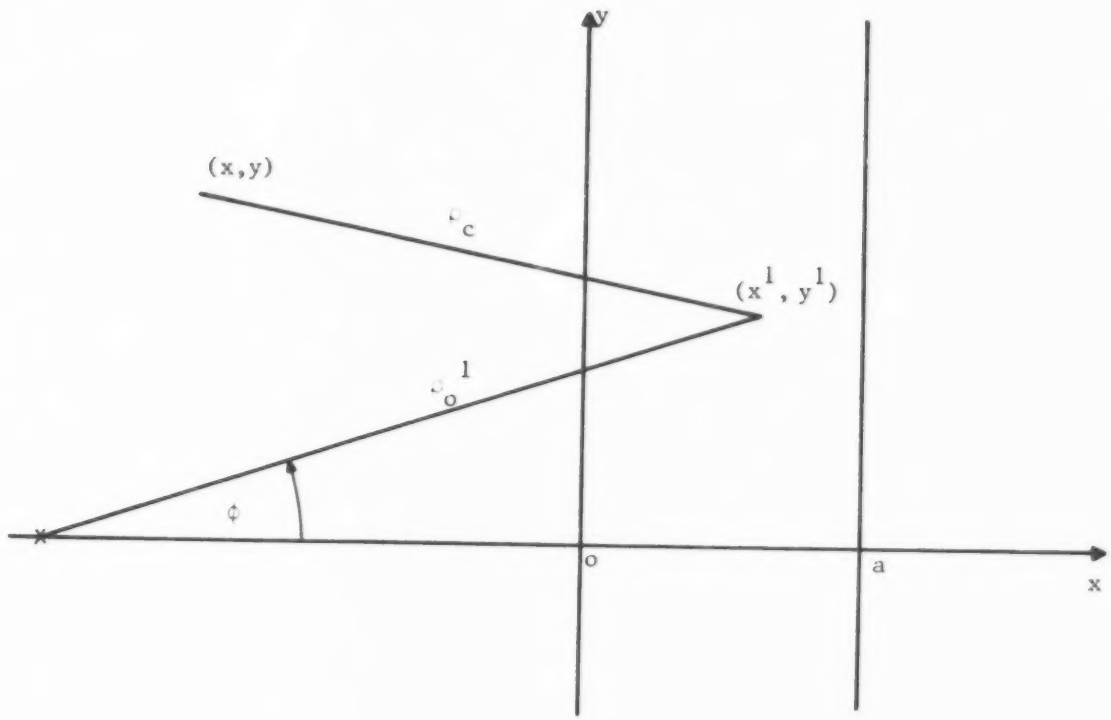


Figure 11 Illustration for Equation (196)

Again we consider the monostatic case, i.e., $x = -x_o$, $y = 0$, then (197) becomes

$$\int_{-\pi/2}^{\pi/2}\int_{p}^{q} e^{-j2k\rho_o'}\left\{\frac{F_3^2(\rho_o')}{\rho_o'} + (\Delta_3 - \Delta_2)\left(\frac{jk}{2\pi}\right)^{1/2}\frac{F_3(\rho_o')}{(\rho_o')^{1/2}}\int_{p}^{\rho_o'} \cdot \frac{F_2(\rho_o' - \rho')}{(\rho_o' - \rho')^{1/2}}\frac{F_3(\rho')}{(\rho')^{1/2}}\,d\rho'\right\}d\rho_o'\,d\phi. \quad (198)$$

We now write the inner integral in (198) as

$$\int_p^q e^{-j2k\rho_o'} G(\rho_o')\,d\rho_o' = \int_p^\infty e^{-j2k\rho_o'} G(\rho_o')\,d\rho_o' - \int_q^\infty e^{-j2k\rho_o'} G(\rho_o')\,d\rho_o' \quad . \quad (199)$$

The first integral in (199) may be handled as per (187). Now the function $G(\rho_o')$ may be expanded for $\rho_o' > q$ as

$$G(\rho_o') \simeq G(q) + G'(q)(\rho_o' - q) + \frac{G''(q)}{2}(\rho_o' - q)^2 + \ldots \tag{200}$$

Hence, using (200) and (193), the leading term in the asymptotic expression of (180) is given by

$$\int_p^q e^{-2jk\rho_o'} (G(\rho_o')\, d\rho_o'$$

$$- \frac{1}{jk}\Big[e^{-2jkp} M(\Delta_3 - \Delta_2, \Delta_2) \frac{F_3^2(p)}{p} - e^{-2jkq}\Big\{\frac{F_3^2(q)}{q}$$

$$+ (\Delta_3 - \Delta_2)\left(\frac{jk}{2\pi}\right)^{1/2} \frac{F_3(q)}{(q)^{1/2}} \int_p^q \frac{F_2(q - \rho')}{(q - \rho')^{1/2}} \frac{F_3(\rho')}{(\rho')^{1/2}}\, d\rho'\Big\}\Big]. \tag{201}$$

Substituting (201) into (196) and performing a stationary phase integration with respect to ϕ yields for (196),

$$\left[(1 - h_x)E_z^{z+}\right]_b \simeq \frac{I\Delta\ell k^2}{j\omega\varepsilon_o(2\pi)} \frac{(\Delta_3 - \Delta_2)}{2\pi}\Big[\left(\frac{\lambda_o x_o}{2}\right)^{1/2} M(\Delta_3 - \Delta_2, \Delta_2)$$

$$\cdot \frac{F_3^2(x_o)}{x_o^2} e^{-j(2kx_o + \frac{\pi}{4})}$$

$$- \left(\frac{\lambda_o(x_o + a)}{2}\right)^{1/2} \Big\{\frac{F_3^2(x_o + a)}{(x_o + a)^2} + (\Delta_3 - \Delta_2)\left(\frac{jk}{2\pi}\right)^{1/2} \frac{F_3(x_o + a)}{(x_o + a)^{3/2}}$$

$$\cdot \int_{x_o}^{x_o + a} \frac{F_2((x_o + a) - \rho')}{((x_o + a) - \rho')^{1/2}} \frac{F_3(\rho')}{(\rho')^{1/2}}\, d\rho'\Big\}\, e^{-j(2k(x_o + a) + \frac{\pi}{4})}\Big] \tag{202}$$

4.3 The Backscattered Field for the Case of a Discontinuity Involving a Finite Region, the Monostatic Case.

Refer now to the term involving the double integral in (183), i.e.,

$$\left[(1 - h_R)E_z^{z+}\right]_b = \frac{I\Delta\ell k^2}{j\omega\varepsilon_o(2\pi)} \frac{jk(\Delta_2 - \Delta_1)}{2\pi} \iint_R F_2(\rho_o') \frac{e^{-jk\rho_o'}}{\rho_o'} \left\{ F_2(\rho_o') \frac{e^{-jk\rho_o'}}{\rho_o'} + \frac{jk(\Delta_2 - \Delta_1)}{2\pi} \iint_R F_1(\rho_a) F_2(\rho_o'') \frac{e^{-jk(\rho_a + \rho_o'')}}{\rho_a \rho_o''} dx''dy'' \right\} dx'dy' \tag{203}$$

where

$$\rho_o' = ((x_o + x')^2 + y'^2)^{1/2}$$

$$\rho_o'' = ((x_o + x'')^2 + y''^2)^{1/2}$$

$$\rho_a = ((x' - x'')^2 + (y' - y'')^2)^{1/2},$$

where again we have placed the observation point at the source location. If we perform the stationary phase integration as in the previous sections then

$$\frac{jk(\Delta_2 - \Delta_1)}{2\pi} \iint_R F_1(\rho_a) F_2(\rho_o'') \frac{e^{-jk(\rho_a + \rho_o'')}}{\rho_a \rho_o''} dx''dy''$$

$$\simeq (\Delta_2 - \Delta_1) \left(\frac{jk}{2\pi\rho_o'}\right)^{1/2} e^{-jk\rho_o'} \int_{\rho_1}^{\rho_2} \frac{F_1(\rho_o' - \rho')}{(\rho_o' - \rho')^{1/2}} \frac{F_2(\rho')}{(\rho')^{1/2}} d\rho' \quad . \tag{204}$$

Using (204) and changing the variables of integration to (ρ_o', ϕ), (203) may be written as

$$\left[(1 - h_R)E_z^{z+}\right]_b = \frac{I\Delta\ell k^2}{j\omega\varepsilon_o(2\pi)} \frac{jk(\Delta_2 - \Delta_1)}{2\pi} \int_\phi \int_{\rho_1(\phi)}^{\rho_2(\phi)} e^{-2jk\rho_o'} G(\rho_o', \phi) d\rho_o' d\phi, \tag{205}$$

where

$$G(\rho_o') = \frac{F_2^2(\rho_o')}{\rho_o'} + (\Delta_2 - \Delta_1)\left(\frac{jk}{2\pi}\right)^{1/2} \frac{F_2(\rho_o')}{(\rho_o')^{1/2}} \int_{\rho_1(\phi)}^{\rho_o'} \frac{F_1(\rho_o' - \rho')}{(\rho_o' - \rho')^{1/2}} \frac{F_2(\rho')}{(\rho')^{1/2}} \, d\rho'.$$

The various terms in (205) are illustrated in Figure 12.

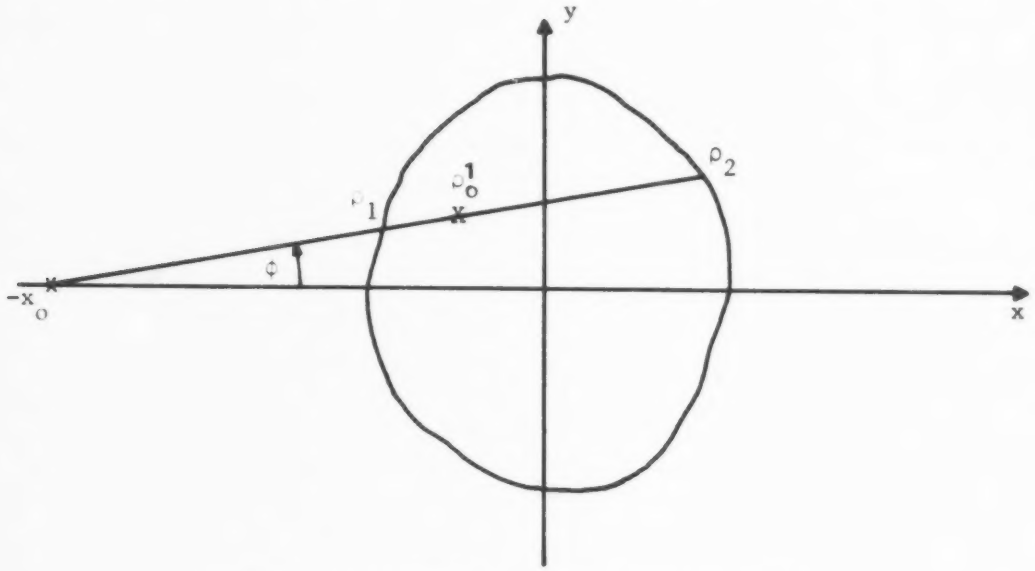


Figure 12 Illustration for Equation (205)

According to (201) then the leading term in an asymptotic expansion of (205) is

$$\left[(1 - h_R)E_z^{z+}\right]_b \sim \frac{I\Delta\ell k^2}{j\omega\epsilon_o(2\pi)} \frac{\Delta_2 - \Delta_1}{2\pi} \int_\phi \Big[F_2^2(\rho_1) \frac{e^{-2jk\rho_1}}{\rho_1} M(\Delta_2 - \Delta_1, \Delta_1)$$

$$- e^{-2jk\rho_2} \Big\{\frac{F_2^2(\rho_2)}{\rho_2} + (\Delta_2 - \Delta_1)\left(\frac{jk}{2\pi}\right)^{1/2}$$

$$\cdot \frac{F_2(\rho_2)}{(\rho_2)^{1/2}} \int_{\rho_1}^{\rho_2} \frac{F_1(\rho_2 - \rho')}{(\rho_2 - \rho')^{1/2}} \frac{F_2(\rho')}{(\rho')^{1/2}} \, d\rho' \Big\}\Big] \, d\phi, \quad (206)$$

where $M(\Delta_2 - \Delta_1, \Delta_1)$ is the function defined in (193). The leading

term in the asymptotic expansion for the integral appearing in (206) may be found by an integration by parts. We have

$$\int_{\rho_1}^{\rho_2} \frac{F_1(\rho_2 - \rho_1)}{(\rho_2 - \rho_1)^{1/2}} \frac{F_2(\rho')}{(\rho')^{1/2}} d\rho' = \int_0^{\rho_2-\rho_1} \frac{F_1(\rho')}{(\rho')^{1/2}} \frac{F_2(\rho_2 - \rho')}{(\rho_2 - \rho')^{1/2}} d\rho'.$$

Then by parts

$$\int_o^{\rho_2-\rho_1} \frac{F_1(\rho')}{(\rho')^{1/2}} \frac{F_2(\rho_2 - \rho')}{(\rho_2 - \rho')^{1/2}} d\rho' = \frac{F_2(\rho_2 - \rho')}{(\rho_2 - \rho')^{1/2}} \int \frac{F_1(\rho')}{(\rho')^{1/2}} d\rho' \Big|_0^{\rho_2-\rho_1}$$

$$- \int_0^{\rho_2-\rho_1} \{ \int \frac{F_1(\rho')}{(\rho')^{1/2}} d\rho' \} [\frac{F_2(\rho_2 - \rho')}{(\rho_2 - \rho')^{1/2}}]' d\rho'. \tag{207}$$

Now since

$$\frac{F_1(\rho')}{(\rho')^{1/2}} = \frac{1}{(\rho')^{1/2}} - j (\pi c_1)^{1/2} e^{-c_1\rho'} \operatorname{erfc} (j(c_1\rho')^{1/2})$$

$$c_1 = -j \frac{k\Delta_1^2}{2} ,$$

we have

$$\int \frac{F_1(\rho')}{(\rho')^{1/2}} d\rho' = j \frac{(\pi c_1)^{1/2}}{c_1} e^{-c_1\rho'} \operatorname{erfc} (j(c_1\rho')^{1/2})$$

and the first term in (207) is

$$j \frac{(\pi c_1)^{1/2}}{c_1} [\frac{F_2(\rho_1)}{(\rho_1)^{1/2}} e^{-c_1(\rho_2 - \rho_1)} \operatorname{erfc} (j(c_1(\rho_2 - \rho_1))^{1/2}) - \frac{F_2(\rho_2)}{(\rho_2)^{1/2}}]. \tag{208}$$

Using (208) in (206) we obtain

$$\left[(1 - h_R)E_z^{z+}\right]_b - \frac{I\Delta\ell k^2}{j\omega\epsilon_o(2\pi)} \frac{\Delta_2 - \Delta_1}{2\pi} \int_\phi \left[\frac{F_2^2(\rho')}{\rho'} e^{-2jk\rho_1} M(\Delta_2 - \Delta_1, \Delta_1)\right.$$

$$\left. - \frac{F_2^2(\rho_2)}{\rho_2} e^{-2jk\rho_2} \left\{\frac{\Delta_2}{\Delta_1} + (1 - \frac{\Delta_2}{\Delta_1}) Q_1(\rho_2 - \rho_1)(\frac{\rho_2}{\rho_1})^{1/2} \frac{F_2(\rho_1)}{F_2(\rho_2)}\right\}\right] d\phi \tag{209}$$

where

$$Q_1(\rho_2 - \rho_1) = e^{-c_1(\rho_2 - \rho_1)} \text{ erfc } (j(c_1(\rho_2 - \rho_1)^{1/2})$$

$$M(\Delta_2 - \Delta_1, \Delta_1) = 1 + \frac{\Delta_2 - \Delta_1}{(2)^{1/2} - \Delta_1} \quad .$$

4.4 Radar Cross Sections

In this section we shall give expressions for the radar range equation and radar cross sections based on the results of sections 4.1 - 4.3.

4.4.1 Radar Range Equation and Radar Cross Section for a Simple Discontinuity, the Monostatic Case

Consider (195) the backscattered field for the simple discontinuity, which we rewrite as

$$E_{zb} = \frac{C}{4\pi} \frac{F_2^2(x_0)}{x_0^2} e^{-j(2kx_0 + \pi/4)} (\frac{\lambda_0 x_0}{2})^{1/2}$$

$$\cdot \frac{(\Delta_2 - \Delta_1)}{\pi} (1 + \frac{\Delta_2 - \Delta_1}{\sqrt{2} - \Delta_1}) \quad , \tag{210}$$

where

$$C = \frac{I\Delta\ell k^2}{j\omega\epsilon_0} = -j\omega\mu_0 I\Delta\ell \quad .$$

Recalling that the transmitting antenna is an infinitesimal vertical dipole, we observe that the free-space vertical field for such an antenna in the horizontal plane is given by

$$E_{z0} = \frac{C}{4\pi} \frac{e^{-jk\rho}}{\rho} \tag{211}$$

where ρ is the radial distance from the antenna. We can also use the standard definition of antenna gain in the horizontal direction for this antenna (see for example Jasik 1961).

$$G_t = \frac{2\pi |E_{z0}|^2}{\eta_0 P_t} \tag{212}$$

where

η_0 = intrinsic impedance of vacuum

P_t = transmitted power .

Applying (210) to (212)

$$G_t = \frac{|C|^2}{8\pi \eta_0 P_t}$$

or

$$\frac{|C|^2}{4\pi} = 2G_t \eta_0 P_t \quad . \tag{213}$$

Referring now to (210), we have

$$|E_{zb}|^2 = \frac{|C|^2}{(4\pi)^2} \frac{F_2(x_0)|^4}{x_0^4} \frac{\lambda_0 x_0}{2\pi^2} |(\Delta_2 - \Delta_1)(1 + \frac{\Delta_2 - \Delta_1}{\sqrt{2} - \Delta_1})|^2 \quad . \tag{214}$$

Let the receiving antenna have effective area A_r, then the received power is:

$$P_r = A_r \frac{|E_{zb}|^2}{2\eta_0} = \frac{\lambda_0^2 G_R}{4\pi} \frac{|E_{zb}|^2}{2\eta_0} , \tag{215}$$

where

G_R = gain of receiving antenna.

Inserting (214) into (215) and rearranging, (215) may be written in the form:

$$P_r = \frac{\lambda_0^2 G_R G_t P_t |F_2(x_0)|^4}{(4\pi)^3 x_0^4} \sigma \tag{216}$$

where

$$\sigma = \frac{2\lambda_0 x_0}{\pi} \left| (\Delta_2 - \Delta_1)(1 + \frac{\Delta_2 - \Delta_1}{\sqrt{2} - \Delta_1}) \right|^2 . \tag{217}$$

It may be seen that (216) is in the form of a standard radar range equation with target cross-section given by σ, i.e., the cross-section of the discontinuity is given by (217).

4.4.2 Radar Range Equation and Radar Cross Section for Two Simple Discontinuities: The Monostatic Case

Refer to (202). Following the same series of steps outlined in the previous section it is straightforward to show that the returned power in this case may be written in the form

$$P_r = \frac{\lambda_0^2 G_R G_t P_t |F_3(x_0)|^4}{(4\pi)^3 x_0^4} \sigma \tag{218}$$

where now

$$\sigma = \frac{2\lambda_0 x_0}{\pi} |(\Delta_3 - \Delta_2)|^2 \left[(1 + \frac{\Delta_3 - \Delta_2}{\sqrt{2} - \Delta_2}) - (\frac{x_0 + a}{x_0})^{1/2} \frac{x_0^2}{F_3^2(x_0)} \right.$$

$$\left\{ \frac{F_3^2(x_0+a)}{(x_0+a)^2} + (\Delta_3 - \Delta_2)(\frac{jk}{2\pi})^{1/2} \frac{F_3(x_0 + a)}{(x_0 + a)^{3/2}} \cdot \right.$$

$$\left. \left. \int_{x_0}^{x_0+a} \frac{F_2((x_0+a)-\rho')}{((x_0+a)-\rho')^{1/2}} \frac{F_3(\rho')}{(\rho')^{1/2}} d\rho' \right\} e^{-j(k2a1+\pi/4)} \right]|^2 . \qquad (219)$$

Various simplifications and approximations are possible in (219), for example, the term involving the integral may be evaluated as in section 4.3.

4.4.3 Radar Range Equation and Radar Cross Section for the Case of a Discontinuity Involving a Finite Region: The Monostatic Case

Refer to (209). Following the same procedure as in 4.4.1 we obtain for the received power

$$P_r = \frac{\lambda_0^2 G_R G_t P_t |F_2(x_0)|^4}{(4\pi)^3 x_0^4} \sigma \qquad (220)$$

where

$$\sigma = \frac{4}{\pi} |\Delta_2 - \Delta_1|^2 \left| \frac{x_0^2}{F_2^2(x_0)} \int_\phi e^{-2jk\rho'} \left[\frac{F_2^2(\rho')}{(\rho')} (1 + \frac{\Delta_2 - \Delta_1}{\sqrt{2} - \Delta_1}) \right. \right.$$

$$- \frac{F_2^2(\rho_2)}{\rho_2} e^{-2jk(\rho_2-\rho_1)} \left\{ \frac{\Delta_2}{\Delta_1} + (1 - \frac{\Delta_2}{\Delta_1}) \right.$$

$$\left. \left. \left. Q_1(\rho_2-\rho_1)(\frac{\rho_2}{\rho_1})^{1/2} \frac{F_2(\rho_1)}{F_2(\rho_2)} \right\} \right] d\phi \right|^2 . \qquad (221)$$

The various distances and angle in (220) and (221) are illustrated in Figure 12. Also recall

$$F_2(\rho) = 1 - j\sqrt{\pi c_2 \rho}\; e^{-c_2\rho}\; \text{erfc}\; (j\sqrt{c_2 \rho})$$

$$Q_1(\rho) = e^{-c_1\rho}\; \text{erfc}\; (j\sqrt{c_1 \rho})$$

$$c_1 = -j\,\frac{k\Delta_1^{\,2}}{2}$$

$$c_2 = -jk\,\frac{\Delta_2^{\,2}}{2}$$

$$\Delta_1 = \frac{\sqrt{n_{01}^{\,2} - 1}}{n_{01}^{\,2}}$$

$$\Delta_2 = \frac{\sqrt{n_{02}^{\,2} - 1}}{n_{02}^{\,2}}$$

$$n_{01}^{\,2} = \varepsilon_{r1} - j\,\frac{\sigma_1}{\omega\varepsilon_0}$$

$$n_{02}^{\,2} = \varepsilon_{r2} - j\,\frac{\sigma_2}{\omega\varepsilon_0} \quad .$$

The Appendix contains a further discussion of (220) and (221) and their application to the problem of estimating HF surface wave radar backscattered energy from iceberg and other ice feature targets.

5. CONCLUSIONS

5.1 Discussion

The problem formulation as carried out in Chapter 2 is basically the same as employed previously for rough surace propagation and scatter (Walsh, 1980) and layered media propagation (Walsh, 1982), [see also Donnelly (1983), Ryan (1983) and Ryan and Walsh (1983)]. The results up to and including Section 2.4 are general in the sense that they are applicable when the source is any finite antenna system and the lower half-space has arbitrary one-dimensional electrical property variation. It appears straightforward to extend the analyses to arbitrary three-dimensional variation.

The special case of electrical property variation consisting of a simple discontinuity as dealt with in Section 2.5 onwards was chosen not only for simplicity but also for application (e.g. the water-ice interface). It should be noted that the formulation technique of Chapter 2 furnishes the boundary conditions necessary for solution. These boundary conditions evolve naturally from the procedure as auxillary equations relating the unknown quantities at the various interfaces.

The solutions as presented in Chapter 3 for the three cases; 1) a surface with one simple discontinuity, 2) a surface with two simple discontinuities, and 3) a surface with a discontinuity involving a finite region, have been labelled approximate in the report. The reasons for this are two-fold: a) the boundary conditions as devised

in Section 3.1 are approximations and subject to the two conditions stated for equation (81). These boundary conditions are of the same nature as employed by Bremmer (1954) and consistantly in the work of Wait, e.g. (1962, 1970, 1974). b) The leading term only in the formal Neumann series solutions, i.e., equations (103), (152) and (175), has been used. In each of the cases 1) and 2) above however, it has been shown that the expressions for the forward propagated electric field agree with established theory. This fact leads to a certain degree of confidence in the results for the backscattered fields; nevertheless, it would be very interesting to pursue the equation of higher order terms in the formal Neumann series expansions, as along the lines of Section 3.4.

5.2 Applications to Predictions of Returned Power and Radar Cross Section for Sea Ice Targets

The results of Chapter 4 may be used to estimate the radar cross section and returned power for sea ice targets. Equations (216) and (217) of Section 4.4.1 may be used to model the water to ice (or ice to water) interface. Also, they may be used for modelling a transition from one ice type to another. Some sample calculations are carried out in Ryan (1983). Equations (218) and (219) may be used in modelling a transition from water to ice to water (or ice to water to ice). These equations give estimates of the cross section and returned power from such a discontinuity.

The Appendix contains a computer program for numerical evaluation of (220) and (221), i.e., the returned power and radar cross section for the case of a discontinuity involving a finite region. These two

equations are proposed as the initial model for predictions of backscattered energy from iceberg targets using HF surface wave radar. This model will be subjected to experimental testing in the forthcoming iceberg detection experiment in the Strait of Belle Isle, beginning June 20, 1983. Data will be collected from iceberg targets allowing experimental estimates of the radar cross section. Data on the area cross section of the icebergs at the water line will also be collected. The predictions of (220) and (221) will be compared with the experimental observation of radar cross section.

Although we are proposing the use of the results of Section 4.4.3 for predictions of HF radar return from icebergs, as outlined above, it is necessary to make some observations. Estimates calculated from (220) and (221) are based on the return because of surface discontinuity effects only. They do not include return from the iceberg sail or return from expected electrical currents at the water to ice discontinuity below the waterline. These latter effects may be important. Also the approximate boundary conditions used in this report should be kept clearly in mind; iceberg ice has very low electrical conductivity and the strict validity of these boundary conditions is in question. The above considerations will be borne in mind when comparing the model predictions with the experimental results. Further work in this regard may very well be necessary and is planned.

The Appendix contains some numerical examples and further observations regarding the visibility of icebergs to HF radars.

REFERENCES

Bremmer, H. (1954), The extension of Sommerfield's formula for the propagation of radio waves over a flat earth to different conductivities of the soil. Physica's Grav. 20, 441.

Donnelly, R. (1983), Electromagnetic scattering from a vertical halfspace discontinuity: operator decomposition approach. M. Eng. thesis, Memorial University of Newfoundland.

Furutsu, K. (1982), A systematic theory of wave propagation over irregular terrain. Radio Science, Vol. 17, 1037-1050.

Gel'fand, I.M., and Shelov, G.E. (1964), Generalized Functions, Academic ; New York.

Hufford, G.A. (1952), An integral equation approach to the problem of wave propagation over an irregular surface. Quart. Appl. Math., 9, 391-404.

Jasik, H. (editor), (1961), Antenna Engineering Handbook, McGraw Hill; New York.

King, R.J. (1969), Electromagnetic wave propagation over a constant impedance plane. Radio Science, Vol. 4, No. 3, 255-268.

Norton, K.A. (1936), The propagation of radio waves over the surface of the earth and in the upper atmosphere. I. Proc. Inst. Radio Engrs., N.Y. 24, 1367-1387.

Norton, K.A. (1937), The propagation of radio waves over the surface of the earth and in the upper atmosphere. II. Proc. Inst. Radio Engrs., N.Y. 25, 1203-1236.

Ryan, J. (1983), Electromagnetic scattering from a vertical halfspace discontinuity: operator expansion approach. M. Eng. thesis, Memorial University of Newfoundland.

Ryan, J., and Walsh, J. (1983), Electromagnetic scattering from a vertical discontinuity with applications to ice hazard detection. Proc. IEEE AP-S. International Symposium, Univ. of Houston, 469-472.

Srivastava, S.K. (1983), Analysis of HF scattering from an ocean surface: an alternate approach incorporating a dipole source. Ph.D. thesis, Memorial University of Newfoundland.

Srivastava, S.K., and Walsh, J. (1983), An alternative analysis of HF scattering from an ocean surface. Proc. IEEE AP-S. International Symposium, Univ. of Houston, 680-683.

Wait, J.R. (1962), Electromagnetic surface waves, in Advances in Radio Research. 1, 157-170, edited by J.A. Saxton, Academic Press ; New York.

Wait, J.R. (1970) Electromagnetic Waves in Stratified Media. Pergamon, Manwell House :Fairview Park, Elmsford N.Y.

Wait. J.R. (1974), Recent analytical investigations of electromagnetic ground wave propagation over inhomogeneous earth models. Proceedings of the IEEE, Vol. 62, No. 8, 1061-1072.

Walsh, J. (1980), On the theory of electromagnetic propagation across a rough surface and calcuations in the VHF region. Technical report prepared for the Defence Research Establishment Atlantic (DREA), 200 pp., OEIC Memorial University, report NO0232.

Walsh, J. (1982), A general theory of the interaction of electromagnetic waves with isotropic horizontally layered media, and applications to propagation over sea ice. C-CORE technical report, 82-9, 55 pp., OEIC Memorial University.

APPENDIX

Computer Program for the Cauculation of Radar Return from Finite Irregular Discontinuities

This appendix contains the Fortran lising of a computer program to evaluate equation (220),

$$\rho_r = \frac{\lambda_0^2 \, G_R G_t P_t |F_2(x_0)|^4}{(4\pi)^3 x_0^4} \sigma \tag{A1}$$

and the simplified version of (221),

$$\sigma = \frac{4}{\pi} |\Delta_2 - \Delta_1|^2 \left| \int_{y_1}^{y_2} e^{2jk(x_0 - \rho_1)} \left[\left(1 + \frac{\Delta_2 - \Delta_1}{\sqrt{2} - \Delta_1}\right) \right.\right.$$

$$\left.\left. - e^{-2jk(\rho_2 - \rho_1)} \left\{ \frac{\Delta_2}{\Delta_1} + \left(1 - \frac{\Delta_2}{\Delta_1}\right) Q_1(\rho_2 - \rho_1) \right\} \right] dy \right|^2. \tag{A2}$$

Referring to (221), in (A2) the following reasonable assumptions are made:

$$F_2(\rho_1) \simeq F_2(\rho_2) \simeq F_2(x_0),$$

further, in the sense of magnitude,

$$\rho_1 \simeq \rho_2 \simeq x_0.$$

Figure 12 illustrates the various distance in A2.

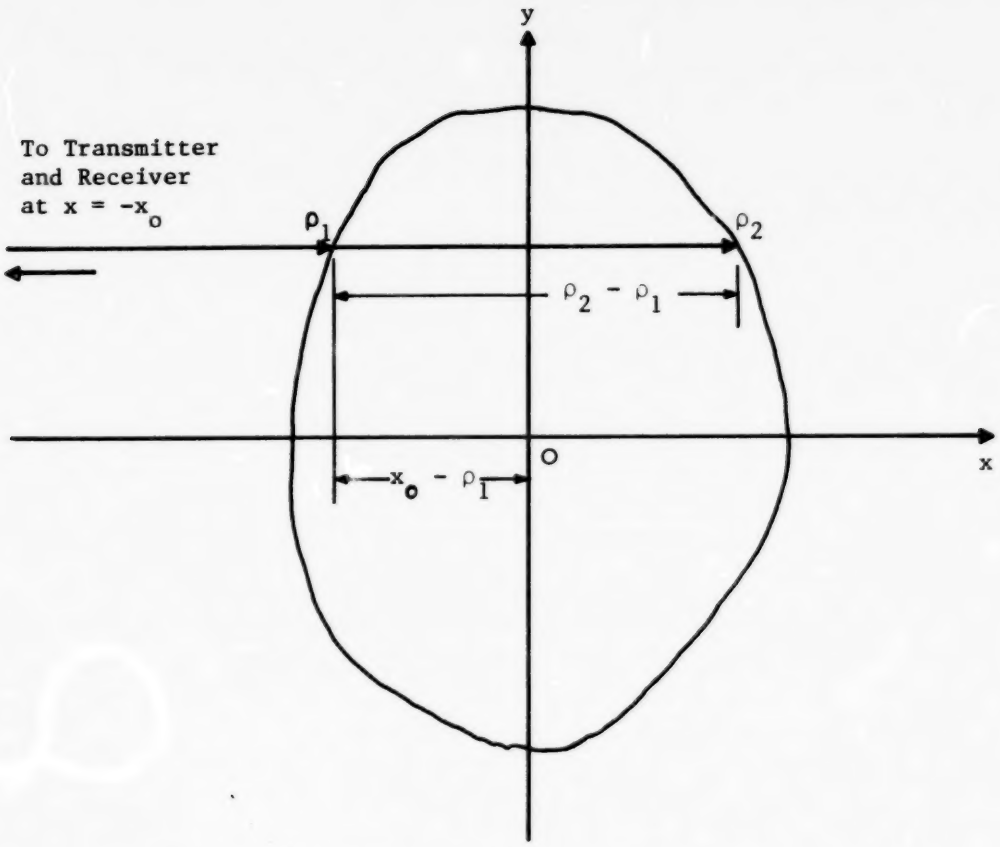


Figure 13. Illustration for (A1) and (A2).

It should be noted that the program contains a function subprogram for the computation of

$$F(\rho) = 1-\sqrt{\pi c\rho}e^{-c\rho}\ \mathrm{erfc}(j\sqrt{c\rho}),\quad c = -jk\,\frac{\Delta^2}{2}\quad ,$$

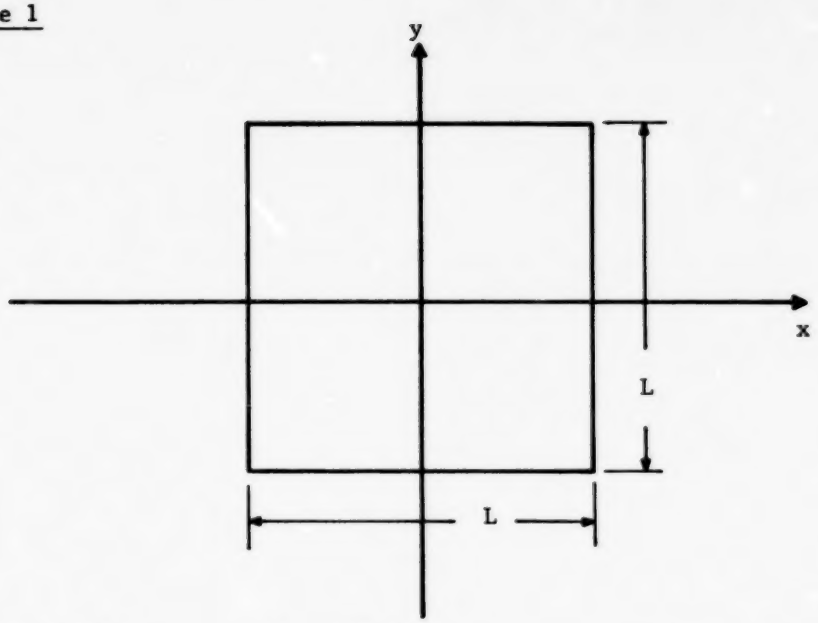
and since

$$Q_1(\rho) = e^{-c_1\rho}\ \mathrm{erfc}\ (j\sqrt{c_1\rho}),$$

then

$$Q_1(\rho) = \frac{1 - F_1(\rho)}{\sqrt{\pi\ c_1\rho}}\quad .$$

Example 1



x_0 = 30 km, f_0 = 25.4 Mhz.

ε_{r2} = 80, ε_{r1} = 5

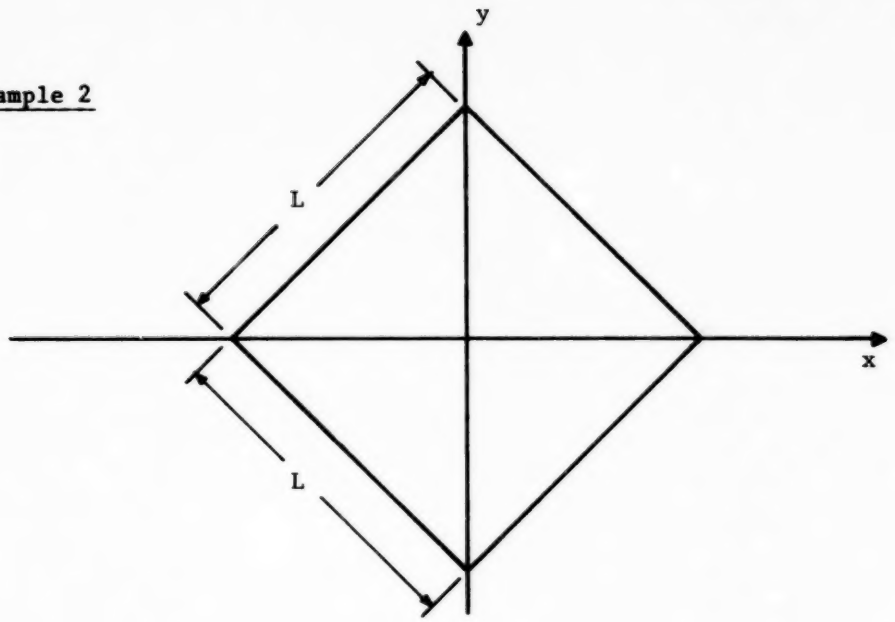
σ_2 = 4, $\sigma_1 = 10^{-5}$

The electrical properties are chosen such that σ_2, ε_{r2} are typical of sea water, while σ_1, ε_{r1} are estimates for the almost fresh water ice of icebergs. The program output for σ is tabulated below. Also calculated is the ratio σ/σ_0 in dB, i.e. 10 log (σ/σ_0), where σ_0 is the cross-section of the first-order sea echo at 30 km for a 8μ-sec pulse and 6° radar antenna beamwidth. The wind direction is assumed to be in the radar look direction and a Phillip's spectrum for fully developed seas is assumed (Srivastava, 1983)

L(m)	$\sigma(m^2)$	10 log (σ/σ_0)
300 (very large)	9.43×10^3	- 6.
200 (large)	4.92×10^3	- 9.
100 (medium)	2.52×10^2	- 21.7
50 (small)	4.41×10^2	- 19.3
10 (bergy bit)	1.4×10^1	- 34.3
5 (growler)	0.91×10^0	- 46.2

$\sigma_0 = 3.77 \times 10^4 \ m^2$

Example 2



x_0 = 30 km $\qquad$ f_0 = 25.4 Mhz

ε_{r2} = 80 $\qquad$ ε_{r1} = 5

σ_2 = 4 $\qquad$ $\sigma_2 = 10^{-5}$

L(m)	$\sigma(m^2)$	10 log (σ/σ_0) (See Ex. 1)
300 (very large)	2.63×10^3	- 11.6
200 (large)	1.01×10^2	- 26.6
100 (medium)	2.02×10^1	- 32.6
50 (small)	1.37×10^2	- 24.2
10 (bergy bit)	2.59×10^0	- 41.5
5 (growler)	$0.270 \times 10^{\circ}$	- 51.3

$\sigma_0 = 3.77 \times 10^4\ m^2$

Accepting the analysis techniques employed in this report, examples 1 and 2 indicate that even relatively small ice features have significant radar cross sections. Since second order sea echo is on the average 30 to 40 dB below first order, and first order is large only in a relatively small band of doppler frequencies. It may be possible to "see" icebergs as small as bergy bits with relatively little data processing. It should be noted, however, that there are certain anomalies in the above results. These can be traced to the fact that (221) and (A2) are sensitive to geometry because of the phase terms. The program gives in addition to σ and P_r the contribution in magnitude and phase from the leading and trailing edges.

```
C*****************************************************************************
C*                                                                           *
C*              ICEBERG CROSS SECTION PROGRAM                                *
C*                                                                           *
C*****************************************************************************

C       ******************
C       DECLARE VARIABLES
C       ******************

        DIMENSION X(100),Y(100),Z(100)
        REAL PR,PT,GT,GR,F0,EPS0,EPS1,EPS2
        REAL SIG1,SIG2,W,SIGMA,K,LAMB,INC,X0
        REAL MSUM,MSUM1,MSUM2,CS,CS1,CS2,RS,RS1,RS2,PSUM,PSUM1,PSUM2
        COMPLEX N1,N2,DELT1,DELT2,CMPLX,CSQRT,CEXP,F,SCEX
        COMPLEX ARG1,ARG2,Q1,NUMD1,NUMD2,C1,A,B,S,FA
        COMPLEX SUM,SUM1,SUM2,FUNCT,FUNCT1,FUNCT2
        INTEGER I,N,J


C       *************************
C       INPUT REQUIRED CONSTANTS
C       *************************

  100   FORMAT(5X,'INPUT TRANSMIT POWER')
  101   FORMAT(5X,'INPUT TRANSMIT AND RECEIVE ANTENNA GAINS')
  102   FORMAT(5X,'INPUT TRANSMIT FREQ(MHz)')
  103   FORMAT(5X,'INPUT EPSILON1 AND SIGMA1')
  104   FORMAT(5X,'INPUT EPSILON2 AND SIGMA2')
  105   FORMAT(5X,'INPUT NUMB OF POINTS AND DELTA Y')
  106   FORMAT(5X,'INPUT POSITION(Y),RHO2-RHO1(Z),AND XO-RHO1 (X)')
  108   FORMAT(5X,'INPUT DISTANCE TO ICE(XO)')
        WRITE(6,100)
        READ(5,*) PT
        WRITE(6,101)
        READ(5,*) GT,GR
        WRITE(6,102)
        READ(5,*) F0
        WRITE(6,108)
        READ(5,*)X0
        WRITE(6,103)
        READ(5,*) EPS1,SIG1
        WRITE(6,104)
        READ(5,*) EPS2,SIG2
```

```
      WRITE(6,105)
      READ(5,*) N,INC

*     ************************
*     INITIALIZE ICEBERG DATA
*     ************************

      WRITE(1,*)N
      DO 10 I=1,N
      WRITE(6,106)
 10   READ(5,*)Y(I),Z(I),X(I)
      WRITE(1,*) (Y(I),Z(I),X(I),I=1,N)
C     CALCULATE CONSTANTS
      EPS0=8.854E-12
      PI=4.0*ATAN(1.0)
      F0=F0*1.0E06
      W=2.0*PI*F0
C     SQRT(MU0*EPSILON0)=3.3356E-09
      K=W*3.3356E-09
      LAMB=2.0*PI/K
      SUM=CMPLX(0.0,0.0)

*     ***************************
*     CALCULATE DELT1 AND DELT2
*     ***************************

      N1=CMPLX(EPS1,-SIG1/(W*EPS0))
      N2=CMPLX(EPS2,-SIG2/(W*EPS0))
      DELT1=(CSQRT(N1-1.0))/N1
      DELT2=(CSQRT(N2-1.0))/N2
      WRITE(6,150)K
 150  FORMAT(5X,'K=',2E20.7)
      WRITE(6,151)DELT1,DELT2
 151  FORMAT(5X,'DELT1=',2E12.7,'DELT2=',2E12.7)

*************************************************
*     INTEGRATION ROUTINE                       *
*************************************************

      S=CMPLX(0.0,1.0)
      DO 20 I=1,N
      ARG1=CMPLX(0.0,2.0*K*X(I))
      ARG2=CMPLX(0.0,-2.0*K*Z(I))
      NUMD1=S*(-K)*(DELT1**2.0)*Z(I)/2.0
      C1=S*(-K)*(DELT1**2.0)/2.0
      B=CSQRT(PI*C1*Z(I))
      A=CMPLX(0.0,1.0)
C
```

```
C
C
C
      Q1=F(NUMD1)
      WRITE(6,199)NUMD1
  199 FORMAT(5X,'NUMD1=',2E20.7)
      Q1=((1-Q1)*(-A))/(B)
C
C
      FUNCT1=SCEX(ARG1)*(1+((DELT2-DELT1)/(SQRT(2.0)-DELT1)))
      SUM1=SUM1+FUNCT1*INC
      FUNCT2=SCEX(ARG1)*SCEX(ARG2)*((1-DELT2/DELT1)*Q1+DELT2/DELT1)
   20 SUM2=SUM2+FUNCT2*INC


*     **********************************
*     OUTPUT CALCULATED CROSS SECTION
*     **********************************

      SUM=SUM1-SUM2
      WRITE(6,1099)SUM1
 1099 FORMAT(5X,'LEADING EDGE',2E20.7)
      WRITE(6,1098)SUM2
 1098 FORMAT(5X,'TRAILING EDGE',2E20.7)
      WRITE(6,1097)SUM
 1097 FORMAT(5X,'TOTAL',2E20.7)
      MSUM1=CABS(SUM1)
      MSUM2=CABS(SUM2)
      MSUM=CABS(SUM)
      CS1=AIMAG(SUM1)
      CS2=AIMAG(SUM2)
      CS=AIMAG(SUM)
      RS1=REAL(SUM1)
      RS2=REAL(SUM2)
      RS=REAL(SUM)
      PSUM=ATAN2(CS,RS)
      PSUM1=ATAN2(CS1,RS1)
      PSUM2=ATAN2(CS2,RS2)
      WRITE(6,1096)MSUM,PSUM
 1096 FORMAT(5X,'MAG OF INTEGRAL=',E15.7,5X,'PHASE=',E15.7)
      WRITE(6,1095)MSUM1,PSUM1
 1095 FORMAT(5X,'MAG LEADING EDGE=',E15.7,5X,'PHASE=',E15.7)
      WRITE(6,1094)MSUM2,PSUM2
 1094 FORMAT(5X,'MAG TRAILING EDGE=',E15.7,5X,'PHASE=',E15.7)
      SIGMA=(4/PI)*(CABS(DELT2-DELT1))**2.0*(CABS(SUM))**2.0
      WRITE(6,198)SIGMA
  198 FORMAT(5X,'SIGMA=',E20.7)
      NUMD2=S*(-K)*(DELT2**2.0)*X0/2.0
      FA=F(NUMD2)
      WRITE(6,799)FA,NUMD2,DELT2
  799 FORMAT(5X,'F2=',2E15.7,'NUMD2=',2E15.7,'DELT2=',2E15.7)
```

```
      F2=(CABS(F(NUMD2)))**4.0
      WRITE(6,120)F2
  120 FORMAT(5X,'THE ABSOLUTE VALUE OF F2**4=',2X,2E20.7)
      PR=((LAMB*2.0)*PT*GT*GR*F2*SIGMA)/(((4.0*PI)**3.0)*(X0**4.0))
      WRITE(6,107) PR
  107 FORMAT(5X,'THE PREDICTED RECEIVED POWER IS:',1X,E20.7)
      END
!
!
!
!
!
      COMPLEX FUNCTION F(P)
**********************************************************************
**                                                                  **
**              ATTENUATION FUNCTION                                **
**                                                                  **
**********************************************************************
      COMPLEX P,R,CEXP,L,CSQRT,SCEX
      REAL MAGP,A,B,C,PI,CABS,CNT
      F=CMPLX(0.0,0.0)
      CNT=1.0
      R=CMPLX(0.0,1.0)
      PI=4.0*ATAN(1.0)
      MAGP=CABS(P)
      IF(MAGP.LE.10.0)GO TO 300
      A=REAL(P)
      B=AIMAG(P)
      C=ATAN2(B,A)
      IF(C.LT.0.0)GO TO 310
C
C     ASYMPTOTIC EXPANSION FOR C.GE.0
C     EXTRA TERM FOR C.GE.0
      F=-2.0*R*CSQRT(PI*P)*SCEX(-P)
C
C
C     CONTINUE ASYMPTOTIC EXPANSION FOR ALL P
  310 L=1.0
      CNT=1.0
  305 L=L*CNT/(2.0*P)
      F=F-L
      CNT=CNT+2.
      T=CABS(L)
      IF(T.GT.1.0E-04)GO TO 305
      GO TO 320
C
C     CONVERGENT SERIES
C
  300 F=1-R*CSQRT(PI*P)*SCEX(-P)
      CNT=1.0
```

```
      L=1.0
  330 L=((-1.0)*L*(2.0*P))/CNT
      F=F+L
      CNT=CNT+2.
      T=CABS(L)
      IF(T.GT.1.0E-04)GO TO 330
  320 RETURN
      END
!
!
!
!
      COMPLEX FUNCTION SCEX(ARG)
C*************************************************************
C**              SCALED COMPLEX EXPONENTIAL               **
C*************************************************************
      COMPLEX ARG,CEXP,CMPLX,CABS
      REAL MAG
      INTEGER ICNT,J
      SCEX=CMPLX(1.0,0.0)
      ICNT=0
      MAG=CABS(ARG)
      IF(MAG.LT.80.0)GO TO 400
  403 ARG=ARG/2.0
      ICNT=ICNT+2
      MAG=CABS(ARG)
      IF(MAG.LT.80.0)GO TO 402
      GO TO 403
  402 CONTINUE
      DO 404 J=1,ICNT
  404 SCEX=SCEX*(CEXP(ARG))
      GO TO 401
  400 SCEX=CEXP(ARG)
  401 RETURN
      END

$
```

POOR

QUALITY

ORIGINAL